Review of syntomic cohomology

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Introduction: We review some parts of the recent foundations of syntomic cohomology, as developed recently by Bhatt, Morrow and Scholze in [BMS19]. It presents deep relations with several important subjects in p-adic arithmetic geometry: integral p-adic Hodge theory [BMS18], [BMS19], the prismatic theory of [BS22], and motivic cohomology [BMS19], [AMMN22], [CMM21].

1 Topological cyclic homology

The following complex of Hochschild homology grew up in algebraic topology, before it showed it was very capable of handling itself in arithmetic.

Definition 1.1. *Let k be a commutative ring, and A a flat algebra over k. The Hochschild complex* $\mathrm{HH}_\bullet(A/k)$ *is defined as*

$$\mathrm{HH}_\bullet(A/k) := A \xleftarrow{b} A \otimes_k A \xleftarrow{b} A \otimes_k A \otimes_k A \xleftarrow{b} \dots$$

with $b : \begin{cases} A^{\otimes_k (n+1)} \to A^{\otimes_k n} \\ a_0 \otimes \cdots \otimes a_n \mapsto a_0 a_1 \otimes a_2 \otimes \cdots \otimes a_n - a_0 \otimes a_1 a_2 \otimes \cdots \otimes a_n + \cdots + \\ \quad (-1)^n a_0 \otimes \cdots \otimes a_{n-1} a_n + (-1)^{n+1} a_n a_0 \otimes \cdots \otimes a_{n-1}. \end{cases}$

One can define variants of Hochschild homology, called cyclic homology HC, negative cyclic homology HC^- and periodic cyclic homology HP. The Hochschild-Kostant-Rosenberg filtration (HKR filtration) is a filtration on these objects, whose graded pieces are related to the cotangent complex (as developed by Quillen, Illusie, ...).

An idea of Waldhausen in the 1980's was to define Hochschild homology and its variants in the world of spectra to get the right "integral" notion (as opposed to "rational"). The notion of spectra is now well-developed (it relies on the ∞-categorical setting of Lurie), as well as Hochschild homology and its variants in this context:

- THH: topological Hochschild homology;
- TC: topological cyclic homology;
- TC^-: negative topological cyclic homology;
- TP: periodic topological cyclic homology.

For instance, one has the following characterisation of TC of Nikolaus-Scholze:

Proposition 1.2. *Let A be a connective $\mathbb{E}_\infty$-ring spectrum. The p-completion $\mathrm{TC}(A;\mathbb{Z}_p)$ of the spectrum $\mathrm{TC}(A)$ satisfies the following natural fiber sequence*

$$\mathrm{TC}(A;\mathbb{Z}_p) \longrightarrow \mathrm{TC}^-(A;\mathbb{Z}_p) \xrightarrow{\varphi_p - \mathrm{can}} \mathrm{TP}(A;\mathbb{Z}_p).$$

Before stating the definition for syntomic cohomology $\mathbb{Z}_p(i)$, we define first define the *quasisyntomic site.*

Definition 1.3 (p-complete (faithful) flatness and Tor amplitude, [BMS19], Definition 4.1)**.** *Let R be a commutative ring. An R-module M is called p-completely flat (resp. p-completely faithfully flat) if $M \otimes^{\mathbb{L}}_R R/p \in \mathcal{D}(R/p)$ is a flat (resp. faithfully flat) R/p-module concentrated in degree zero. Similarly, an object $N \in \mathcal{D}(R)$ has p-complete Tor amplitude in $[a,b]$ if $N \otimes^{\mathbb{L}}_R R/p \in \mathcal{D}(R/p)$ has Tor amplitude in $[a,b]$.*

Remark that, if R is a $\mathbb{F}_p$-algebra, then p-complete flatness (resp. p-complete Tor amplitude in $[a,b]$) is the same as flatness (resp. Tor amplitude in $[a,b]$).

Definition 1.4 (The quasisyntomic site)**.** *(1) A commutative ring R is called* quasisyntomic *if it is p-complete, has bounded p-power torsion (that is, $R[p^\infty] = R[p^n]$ for some integer $n \geqslant 0$), and $\mathbb{L}_{R/\mathbb{Z}_p}$ has p-complete Tor-amplitude in $[-1,0]$. We let* QSyn *be the category of quasisyntomic rings, with all ring homomorphisms (and not just the quasisyntomic ones).*

(2) We call a map $A \to B$ in QSyn *a quasisyntomic map (resp. quasisyntomic cover) if $A \to B$ is p-completely flat (resp. p-completely faithfully flat), and if $\mathbb{L}_{B/A} \in \mathcal{D}(B)$ has p-complete Tor-amplitude in $[-1,0]$. The opposite category of QSyn acquires the structure of a site by declaring covers to be quasisyntomic covers.*

(3) An object $S \in$ QSyn *is* quasiregular semiperfectoid *if S admits a map from a perfectoid ring, and if the Frobenius morphism on S/p is surjective. We denote by* QRSPerd $\subset$ QSyn *the full subcategory spanned by quasiregular semiperfectoid rings.*

Remark that quasiregular semiperfectoid rings arise naturally as tensor products of the form $R_{\mathrm{perf}} \otimes_R \cdots \otimes_R R_{\mathrm{perf}}$, for R a smooth algebra over a perfect field of characteristic p.

Example 1.5. Any perfectoid ring R is a quasiregular semiperfectoid ring. Indeed, the cotangent complex $\mathbb{L}_{R/\mathbb{Z}_p}$ has p-complete Tor-amplitude concentrated in degree -1, and has bounded p-torsion (in fact, $R[p^\infty] = R[p]$).

Definition 1.6. *Let A be a quasisyntomic ring. Denote by $\mathbb{Z}_p(n)(A)$ the graded pieces of the filtration on $\mathrm{TC}(A;\mathbb{Z}_p)$*

$$\mathbb{Z}_p(n)(A) := \mathrm{gr}^n\mathrm{TC}(A;\mathbb{Z}_p)[-2n].$$

The complexes $\mathbb{Z}_p(n)(A)$, indexed by integers n, are called syntomic cohomology *of A.*

2 Prisms and prismatic cohomology

To define prisms, we need the following notion of δ-ring, which is a variant of that of a ring with a lift of Frobenius.

Definition 2.1. *A δ-ring is a pair (A, δ) where A is a commutative ring and $\delta : A \to A$ is a map of sets with $\delta(0) = \delta(1) = 0$, satisfying the following two identities:*

$$\delta(xy) = x^p\delta(y) + y^p\delta(x) + p\delta(x)\delta(y),$$
$$\delta(x + y) = \delta(x) + \delta(y) + \frac{x^p + y^p - (x + y)^p}{p}.$$

Definition 2.2. *A prism is a pair (A, I) where A is a δ-ring (which induces a lift of the Frobenius on A/p, denoted φ_A), and $I \subseteq A$ is an ideal defining a Cartier divisor in* $\mathrm{Spec}(A)$*, satisfying the following two conditions:*

- *The ring A is derived (p, I)-adically complete*[1].
- *The ideal $I + \varphi_A(I)A$ contains p.*

Theorem 2.3 ([BS22], Theorem 1.11 (1))**.** *Let S be a semiperfectoid ring. Then the category $(S)_{\mathbb{\Delta}}$ (underlying the absolute prismatic site $(S)_{\mathbb{\Delta}}$) has an initial object $(\mathbb{\Delta}_S, I)$, and $I = (d)$ is principal.*

One can attach a Nygaard filtration to the object $\mathbb{\Delta}_S$, for S a quasiregular semiperfectoid ring.

Definition 2.4 (Divided Frobenius maps)**.** *Let S be a quasiregular semiperfectoid ring with associated prism $(\mathbb{\Delta}_S, (d))$. For $i \geqslant 0$, the divided Frobenius map φ_i on the (Nygaard-completed) prism $\widehat{\mathbb{\Delta}}_S$ is*

$$\varphi_i := \frac{\varphi}{d^i} : \mathcal{N}^{\geqslant i}\widehat{\mathbb{\Delta}}_S\{i\} \to \widehat{\mathbb{\Delta}}_S\{i\},$$

where φ is the Frobenius map coming from the δ-ring structure of $\mathbb{\Delta}$, and d is a nonzero divisor, and $\{i\}$ is the Breuil-Kisin twist (which can be trivialised by fixing a map from a perfectoid ring to S).

Definition 2.5. *Let S be a quasiregular semiperfectoid ring. Then*

$$\mathbb{Z}_p(i)(S) \simeq \mathrm{hofib}(\varphi_i - 1 : \mathcal{N}^{\geqslant i}\widehat{\mathbb{\Delta}}_S\{i\} \to \widehat{\mathbb{\Delta}}_S\{i\}).$$

This construction forms a sheaf on the site of quasiregular semiperfectoid algebras. One can then globalize to get a sheaf $\mathbb{Z}_p(i)$ on the whole quasisyntomic site (which is called "unfolding" in the litterature).

Theorem 2.6 ([BS22], Theorem 1.15)**.** *Let (A, I) be a bounded prism and let $X = \mathrm{Spf}(R)$ be an affine smooth p-adic formal scheme over A/I. There is a canonical isomorphism:*

$$R\Gamma_{\mathbb{\Delta}}(X/A) \simeq R\Gamma(X_{\mathrm{qsyn}}, \mathbb{\Delta}_{-/A}).$$

[1] Remark that the notion of derived completeness coincide with the usual completeness for *bounded* prisms.

3 Motivic nature

In the 1980's, it was predicted (by Beilinson, Deligne, ...) that a theory of motivic cohomology $\mathrm{H}^n(X, \mathbb{Z}(i))$ for schemes X should exist, where $n \in \mathbb{Z}$ is the cohomological degree, and $i \geqslant 0$ is the motivic weight indexing the so-called Tate twist $\mathbb{Z}(i)$. Motivic cohomology is expected to be the graded pieces of a filtration on algebraic K-theory, and to be related to motives[2]. From this point of view, syntomic cohomology is expected to be p-adic étale motivic cohomology:

Theorem 3.1. *Let X be a smooth scheme over a perfect field k of characteristic p. Then there is an isomorphism of sheaves of complexes in the étale site of X, between syntomic cohomology $\mathbb{Z}_p(n)$ and p-adic étale sheafified motivic cohomology $\mathbb{Z}_p(n)^{\mathrm{mot}}$ (which is constructed in this case by Voevodsky).*

In the ℓ-adic case, étale and usual/Zariski motivic cohomologies are related via the Beilinson-Lichtenbaum conjecture.

Theorem 3.2 (Beilinson–Lichtenbaum conjecture)**.** *Let X be a smooth scheme over a field k in which the prime ℓ is invertible. Then for any $i \geqslant 0$, there is an isomorphism of sheaves of complexes in the Zariski site of X*

$$\mathbb{Z}/\ell\,\mathbb{Z}(i)_{X_{\mathrm{Zar}}} \simeq \tau^{\leqslant i}\mathrm{R}\varepsilon_*(\mu_\ell^{\otimes i})$$

where $\varepsilon : X_{\text{ét}} \to X_{\mathrm{Zar}}$ is the restriction from the étale site to the Zariski site.

In the p-adic setting (in which we are when dealing with syntomic cohomology), the constant étale sheaf $\mu_\ell^{\otimes i}$ has to be replaced by the (étale) logarithmic de Rham-Witt sheaf, and we have the following:

Theorem 3.3 ([BMS19], Theorem 1.15 (1))**.** *Let A be a smooth k-algebra, where k is a perfect field of characteristic p. Then there is an isomorphism of sheaves of complexes on the pro-étale site of $X = \mathrm{Spec}(A)$*

$$\mathbb{Z}_p(i) \simeq W\Omega^i_{X,\log}[-i].$$

4 Concrete computations

One has the following general vanishing result for syntomic cohomology:

Theorem 4.1 ([AMMN22], Theorem G))**.** *Let A be a quasisyntomic ring. Then the syntomic cohomology complex $\mathbb{Z}_p(i)(A)$ is concentrated in cohomological degrees $\leqslant i+1$. Moreover, if A is a strictly henselian local ring, then the complex $\mathbb{Z}_p(i)(A)$ is concentrated in cohomological degrees $\leqslant i$.*

Locally on the quasisyntomic site, one has more precise informations.

Proposition 4.2 ([BMS19])**.** *If S is a quasiregular semiperfectoid ring, then the completed prism $\widehat{\Delta}_S$ is concentrated in degree 0, and thus $\mathbb{Z}_p(i)(S) = \mathrm{hofib}(\varphi - \mathrm{can} : \mathcal{N}^{\geqslant i}\widehat{\Delta}_S \to \widehat{\Delta}_S)$ is concentrated in degree 0 and 1.*

Example 4.3. If $S = R$ is a perfectoid ring, then $\mathbb{Z}_p(i) = \mathrm{hofib}(A_{\mathrm{inf}}(R) \to A_{\mathrm{inf}})$.

In fact, one can be more precise (though it requires more work):

Theorem 4.4 ([BS22])**.** *For each integer $n \geqslant 0$, the sheaf $\mathbb{Z}_p(n)$ is discrete and p-torsionfree.*

Corollary 4.5. *Locally on the quasisyntomic site, p-completed K-theory is concentrated in even degrees.*

[2]The motivic cohomology groups $\mathrm{H}^n(X, \mathbb{Z}(i))$ should be, from the point of view of motives, the Ext groups between X and the i-th Tate twist in the abelian category of motives.

Bibliography

[AMMN22] Benjamin Antieau et al. "On the Beilinson fiber square". In: *Duke Math. J.* (2022). to appear.

[BLM21] Bhargav Bhatt, Jacob Lurie, and Akhil Mathew. "Revisiting the de Rham–Witt complex". In: *Astérisque* 424 (2021), pp. viii+165.

[BMS18] Bhargav Bhatt, Matthew Morrow, and Peter Scholze. "Integral p-adic Hodge theory". In: *Publ. Math. Inst. Hautes Études Sci.* 128 (2018), pp. 219–397.

[BMS19] Bhargav Bhatt, Matthew Morrow, and Peter Scholze. "Topological Hochschild homology and integral p-adic Hodge theory". In: *Publ. Math. Inst. Hautes Études Sci.* 129 (2019), pp. 199–310.

[BS22] Bhargav Bhatt and Peter Scholze. "Prisms and prismatic cohomology". In: *Ann. of Math. (2)* 196.3 (2022), pp. 1135–1275.

[CMM21] Dustin Clausen, Akhil Mathew, and Matthew Morrow. "K-theory and topological cyclic homology of henselian pairs". In: *J. Amer. Math. Soc.* 34.2 (2021), pp. 411–473.

[Fri05] Eric M. Friedlander. "Motivic complexes of Suslin and Voevodsky". In: *Handbook of K-theory. Vol. 1, 2.* Springer, Berlin, 2005, pp. 1081–1104.

[HN20] Lars Hesselholt and Thomas Nikolaus. "Topological cyclic homology". In: *Handbook of homotopy theory.* CRC Press/Chapman Hall Handb. Math. Ser. CRC Press, Boca Raton, FL, 2020, pp. 619–656.

[Lod92] Jean-Louis Loday. *Cyclic homology.* Vol. 301. Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Appendix E by María O. Ronco. Springer-Verlag, Berlin, 1992, pp. xviii+454.

[Mor19] Matthew Morrow. "Topological Hochschild homology in arithmetic geometry". `swc-math.github.io/aws/2019/2019MorrowNotes.pdf`. 2019.