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par

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Contents

Acknowledgements

Foreword and manual

The point of this document is to present the research I have done since the defence of my PhD thesis. One of the first questions I had to consider then concerned the properties of the joint law of variations of quantities that happened to model heat fluxes, but also had the bad taste of not commuting with one another. Having just been made a doctor for work on quantum probability, I knew very well that such a joint law didn't exist, but still wanted to give it a meaning.

It took me a few years of non-commutative work to exhibit relevant probabilistic objects in the above problem. The pleasure of surmounting mathematical difficulties, however, was obscured by my doubts about the physical meaning of the results. It took me a few more years to understand that a new idea, now commonly called "two-time measurement", could give a genuine probabilistic formulation to the problem at hand, with a much clearer physical meaning, and led to statistical formulations of the basic principles of thermodynamics for quantum systems. Of course there was a price to pay for such gains: these basic principles are about variations of quantities such as heat or entropy, and one had to change the definition of these variations, but from a physical point of view these new definitions were actually an improvement. These two steps encouraged me to use exclusively classical probability in approaching problems of quantum statistical mechanics. In parallel, the surge in interest for repeated (indirect) measurement of quantum systems, motivated in particular by new experiments allowing to manipulate individual particles (as the experiments in cavity quantum electrodynamics, or with trapped ions, which earned Haroche and Wineland their Nobel prize) offered me another set of problems formulated in probabilistic language, regarding both the measurement outcomes, and their (random) back action on the observed system. Last, the study of another thermodynamical principle, that takes the form of an inequality expected to be saturated in the limit of infinitely slow transformations, led me to consider time-dependent systems in the so-called adiabatic limit. The above four topics: non-commutative approach to variations, two-time measurements, repeated measurements, adiabatic systems, constitute the four chapters of the present document.

Obviously, and despite the above case for using classical probability, non-commutativity plays an essential part in my work. Indeed, the dynamical systems of interest are described by the quantum formalism, using observables, states and evolution groups. This formalism is therefore involved both in the definition of the (classical) probability distributions I study, but also in the expression of some of the physical laws of interest, which often relate the properties of the distributions to those of the dynamical system. In the simplest cases, the quantum objects are essentially vectors and matrices, and then I mention them explicitly. When investigating thermodynamical properties, however, one is often led to consider systems involving "reservoirs" described by more general operator algebras. Some of my works actually consider specifically systems described by such operator algebras. Discussing such systems would have taken me too far from the main probabilistic theme of this document and I have therefore chosen not to treat them in detail. This means that some of my articles will not be mentioned here; this also means that I will sometimes try awkwarkdly to explain how, had some object been introduced, we could prove a formula that involves it, and then go on to explain both object and formula by analogy. In doing so, I hope not to dissatisfy both readers who know quantum dynamical systems, and readers who don't.

Let us add a few lines as a reading manual: this document comprises a chapter 0 that both introduces and summarizes subsequent chapters 1 to 4, which give more detail on my works. Reading chapter 0 is obviously recommended to the busy reader, but even readers aiming to read further chapters will have to start there, as this is where some notions are defined. References such as [P8] relate to publications to which I have contributed, and their list begins on page 4. Of those publications, references [P1] to [P5] are part of my PhD research. References such as [108] relate to publications to which I have not contributed, and their list begins on page 12. Theorems, propositions and other statements are numbered according to the chapters where they appear, Theorem 1.11 for example being found in chapter 1. Sections and equations do not carry a chapter number, section 4 of chapter 2 being referred to as section 4 in that same chapter and as section 2.4 in other chapters.

Chapter 0

Introduction and summary

In the present chapter, we introduce and motivate the notions of interest for the rest of this document, and summarize our results. We start by recalling the basics of the orthodox model of quantum mechanics, then, in sections 1 à 4, describe the content of the chapters bearing the same numbers. We work essentially on Hilbert spaces of finite dimension, but try to put forward the objects relevant to the general case, and this will lead us to make choices of notation which might seem pedantic or unncessarily complicated. In particular, every time that we mention regularity assumptions in this introduction, such hypotheses will be automatically verified in finite-dimensional cases.

The quantum formalism

Here we recall the basics of the Dirac-von Neumann formalism for quantum mechanics, and refer the reader to reference [44,76] for a mathematical description or to [116] for more physics. In this framework, the state space of a quantum system is represented by a Hilbert space $\mathcal{H}$, of which we denote $\langle \cdot, \cdot \rangle$ the scalar product (which we always assume is linear in the second variable). One considers (for the moment) that any physical quantity of the system is represented by a self-adjoint element of an algebra $\mathcal{O}$ of operators on $\mathcal{H}$, which will be unless otherwise mentioned $\mathcal{O} = \mathcal{B}(\mathcal{H})$; these self-adjoint elements of $\mathcal{O}$ will therefore be called *observables*. For us, the state of a system will be represented by a linear form on $\mathcal{O}$ that is positive (in the sense that it maps positive semidefinite operators to nonnegative reals), maps the identity Id to 1, and verifies regulary properties analogous to the monotone convergence property of probability theory. When $\mathcal{O} = \mathcal{B}(\mathcal{H})$, any such linear form can be written as $X \mapsto \operatorname{tr}(\rho X)$ with ρ a trace-class positive semidefinite operator with trace 1, called a *density matrix*, but we will indifferently call *state*, and denote by the same symbol, both the linear form and the density matrix. The set of density matrices on $\mathcal{H}$ will be denoted $\mathcal{S}(\mathcal{H})$, and is a convex set of the ideal $\mathcal{I}_1(\mathcal{H}_\mathcal{S})$ of trace-class operators. A *pure state* will be a state of the form $X \mapsto \langle \phi, X\phi \rangle$ with $\phi \in \mathcal{H}_\mathcal{S}$ a norm-one vector. The associated density matrix is then the projector on $\mathbb{C}\phi$, which is $|\phi\rangle\langle\phi|$ when written in the Dirac convention that denotes for ψ, ϕ in $\mathcal{H}$, by $|\psi\rangle\langle\phi|$ the map $v \mapsto \langle \phi, v \rangle \psi$.

If the system is closed (in the physical sense), its dynamics is described by a group $(\tau^t)_t$ of automorphisms of $\mathcal{B}(\mathcal{H}_\mathcal{S})$, that, if it is strongly continuous, is necessarily of the form $\tau^t(X) = \mathrm{e}^{+\mathrm{i}tH} X \mathrm{e}^{-\mathrm{i}tH}$ with H an observable called the *Hamiltonian* of the system. This description of the dynamics in which the observables evolve according to the flow $X \rightsquigarrow \tau^t(X) =: X_t$ and state do not change (called the Heisenberg picture) is equivalent to the description in which observables do not change, and states evolve according to $\rho \rightsquigarrow \rho \circ \tau^t =: \rho_t$ (called the Schrödinger picture), the equivalence being given by $\rho_t(X) = \rho(X_t)$.

The last element of the formalism concerns measurements, or more precisely projective measurements, also called von Neumann measurements. If one measures an observable X when the system is in the state ρ, the *Born rule* states that the outcome is random and the possible results are the elements of the spectrum of X. To give an expression for the associated probabilities, let us start with the case where $\mathcal{H}$ is finite-dimensional; in that case X, being self-adjoint, can be diagonalized as

$$X = \sum_{x \in \mathrm{sp}\, X} x\, \pi_X(x)$$

where the $\pi_X(x)$ are orthogonal projectors. Then the probability of observing an outcome contained in a Borel set E is

$$\mathrm{tr}\,\big(\rho\, \pi_X(E)\big) = \sum_{x \in E} \mathrm{tr}\,\big(\rho \pi_X(x)\big) \text{ where } \pi_X(E) = \sum_{x \in E} \pi_X(x)$$

and, conditionally on the fact that the outcome is in E, one has to consider (this is the projection postulate, that we will not comment on) that the state of the system after the measurement is

$$\frac{\pi_X(E) \rho\, \pi_X(E)}{\mathrm{tr}\,\big(\rho \pi_X(E)\big)}.$$

To describe the general case for the measurement of an observable $X \in \mathcal{O}$ when the system is in the state ρ, let us recall that von Neumann's spectral theorem (see e.g. Theorem VIII.6 in [44]) tells us that X can be written as

$$X = \int x\, \mathrm{d}\pi_X(x)$$

where π_X is a projection-valued measure, that is, a map from the set of Borel sets of $\mathbb{R}$ to the set of orthogonal projectors of $\mathcal{H}$, that is σ-additive and maps $\mathbb{R}$ to the identity Id. Then the probability to observe a result contained in a Borel set E of $\mathbb{R}$ is simply $\rho \circ \pi_X(E)$. One can define a functional calculus for bounded Borel functions by giving a definite meaning to $f(X) := \int f(x)\, \mathrm{d}\pi_X(x)$, which then satisfies $\rho\big(f(X)\big) = \int f(x)\, \mathrm{d}(\rho \circ \pi_X)(x)$, and $\rho \circ \pi_X$ is a probability distribution; we will say that *the distribution of X in the state ρ is $\rho \circ \pi_X$*. The expectation of that distribution will then be $\int x\, \mathrm{d}(\rho \circ \pi_X)(x) = \rho(X)$. In addition, conditionally on observing an outcome in E, one must consider that the state after the measurement becomes

$$X \mapsto \frac{\rho\big(\mathbb{1}_E(X) X \mathbb{1}_E(X)\big)}{\rho\big(\mathbb{1}_E(X)\big)}. \tag{1}$$

Measurements are therefore the point where randomness makes it way to the quantum formalism; they will be the common denominator of all problems considered in chapters 2, 3 and 4.

1. Fluctuation-dissipation and non-commutative central limit

In the canonical situation of out-of-equilibrium statistical mechanics, various infinite systems (generally called "reservoirs") $\mathcal{R}_1,\ldots, \mathcal{R}_\ell$ initially at thermal equilibrium at different temperatures are put in contact, so that the full system settles into a "steady state" ρ_+ that is not an equilibrium state and in particular displays non-trivial heat fluxes: one can then identify observables $\Phi_1,\ldots,\Phi_\ell$ representing these heat fluxes, and one expects the average heat fluxes $\rho_+(\Phi_i)$ to not be identically zero. The general study of these fluxes was long limited (even in the classical case) to very general properties saying essentially that the sum of the average fluxes was zero (i.e. $\sum_{i=1}^{\ell} \rho_+(\Phi_i) = 0$), and that the average Clausius entropy associated to these fluxes was nonnegative (i.e. $\sum_{i=1}^{\ell} \beta_i \rho_+(\Phi_i) \geq 0$; the β_i will be defined below). A simplified situation where the theory is better understood is that of *linear reponse*, where the initial temperatures of the reservoirs are close to one another, and therefore equal up to at most ϵ. The three pillars of linear response theory concern the coefficient of the first order of dependence of the average flux coming out of $\mathcal{R}_i$, i.e. $\rho_+(\Phi_i)$, in the temperature of $\mathcal{R}_j$. These three pillars are the Kubo formula, that expresses these coefficients as dynamical correlations between these fluxes "at equilibrium", that is, when the reservoirs are initially all at the same temperature; Onsager's reciprocity relations, that express a symmetry in those coefficients; and the fluctuation-dissipation theorem, that relates these coefficients to the asymptotic joint laws of the time-fluctuations at equilibrium of the fluxes Φ_i. The articles [P6, P7] investigate the Kubo formulas and Onsager's reciprocity relations for quantum systems; they do not fit in the purely probabilistic picture of the present introduction. Indeed, the Kubo formulas can only be expressed in terms of the dynamical system that describes the whole of the system, which is necessarily "infinite". One therefore needs to consider algebras of observables $\mathcal{O}$ which are not the simple algebras $\mathcal{B}(\mathcal{H})$, and to consider more general states and dynamics than we have done so far. One then enters the framework of C*-algebraic dynamical systems, which we will not discuss here. Let us simply remark that the statement and the proof of the Kubo formula and Onsager relations raise only technical, and not conceptual, difficulties.

This is not the case for the fluctuation-dissipation theorem since it concerns joint laws. Indeed, in the quantum framework, it is not possible to give a satisfactory meaning to the joint law of two noncommuting observables. It is however possible to give a meaning to the probabilities associated with sequential measurements, i.e. when one measures one observable, then the other. More precisely, it is possible to measure first X, then Y; the Born rules mentioned above show that the probability to obtain an outcome in E and then an outcome in F (for E, F two Borel sets) is

$$\rho\big(\mathbb{1}_E(X)\mathbb{1}_F(Y)\mathbb{1}_E(X)\big).$$

This quantity, as a function of $E \times F$, does not in general define a measure on $\mathbb{R}^2$ (it is not additive in E). One can however interpret it, for fixed E and F, as a probability value, and to ask the question of its asymptotic behaviour if X, Y are replaced by families X_t, Y_t that depend on a parameter t.

One can then ask whether there exists a simple criterion, in the fashion of the Lévy–Cramér theorem, proving a large-time convergence of these probabilities for sequential measurements, starting from so-called pseudo-characteristic functions – which could for example be $(\alpha_1, \alpha_2) \mapsto \rho(\mathrm{e}^{\mathrm{i}\alpha_1 X_t}\mathrm{e}^{\mathrm{i}\alpha_2 Y_t})$. These questions are discussed in section 1.1, which describes the results of the reference [P9]. One then identifies, in the case where the X_t, Y_t are fluctuations "normalized in $1/\sqrt{t}$" a generic limiting structure and a criterion allowing to deduce the convergence of pseudo-characteristic functions from that of the simpler characteristic functions; this is described in section 1.2, which discusses the results contained in [P9] and [P10].

One then applies the general approach we just described to give in section 1.3 the results of [P8]: a central limit theorem, in the sense of convergence of probabilities of sequential measurements, for a class of models called "fermionic systems". In addition, to come back to the initial motivation which was linear response theory, when one considers these fluctuations in the state ρ which is the equilibrium state (i.e. when the reservoirs $\mathcal{R}_1$,…,$\mathcal{R}_\ell$ are all at the same temperature), these limits of probabilities of sequential mesurements define a true probability measure on pairs of measures. This proves, at last, a fluctuation-dissipation theorem for those models; this theorem will turn out to be unsatisfactory, but for physical (and not mathematical) reasons.

2. Two-time measurements and statistical formulations of thermodynamics

Obtaining a mathematically rigorous fluctuation-dissipation theorem for fermionic systems does not settle the question of its physical meaning: indeed, the fluctuations of interest are those of the variation of heat in the reservoirs, represented by the change in the Hamiltonians H_i between times 0 and t. In the results described in the previous section, we have represented this variation by the observable $\tau^t(H_i) - H_i$. This is perfectly sound mathematically, but one can wonder what actual experiment could produce a measurement of this observable. In addition, a specific linear combination $S = \sum_{i=1}^{\ell} \beta_i H_i$ should represent the total entropy of the system. Yet if we apply the same method to the entropy S as to the Hamiltonians H_i, then the variation of entropy should be modelled by the observable $\tau^t(S) - S$; but then one can see that the distribution of this observable in e.g. the initial system does not satisfy a fundamental relation called *fluctuation relation* which holds for classical systems. This is discussed in section 2.1.

Alternative definitions of the entropy variation have been suggested, that verify the fluctuation relation: the most satisfactory date from the beginning of the years 2000, and are due to Matsui and Tasaki in [120] on the one hand, to Kurchan in [89] on the other hand. The proposal of Matsui and Tasaki defines the law of the entropy variation as the distribution (in the sense given on page 20) of the self-adjoint operator $\log \Delta_{\rho_t|\rho}$ in (a representation of) the state ρ_t, where $\Delta_{\rho_t|\rho}$ is a modular relative, stemming from the Tomita–Takesaki theory for von Neumann algebras. This definition has the advantage of applying from the start to a wide choice of algebras $\mathcal{O}$; on the other hand, its physical meaning was not clear since $\log \Delta_{\rho_t|\rho}$ does not belong to the set $\mathcal{O}$ of observables (nor to its standard representation – see remark 2.9). The proposal of Kurchan defines the entropy variation between 0 and t as the difference between a first measurement, at time $t = 0$, of an "entropy observable " that belongs to $\mathcal{O}$, and a second measurement of that same observable but after the system was perturbed by the first measurement and evolved during a time t. This proposal therefore has a well-defined physical meaning, even if the considered quantity is not derived from a single measurement but from a *two-time measurement*, and defines a notion of trajectory (defined as the successive measurement outcomes). On the other hand, it is essentially limited to the case of finite-dimensional systems. The article [P11] shows that these two proposals are in fact identical for finite systems. One then has a definition of a random variable representing the entropy variation between times 0 and t, that has a satisfactory physical interpretation, and applies to numerous cases: modular theory is indeed robust nough to ensure good convergence properties for approximation schemes where a general algebra is described as the limit of finite-dimensional algebras, and this allows to extend the interpretation of this random variable as the limit of outcomes of two-time measurements. We will discuss these definitions in section 2.2. The fluctuation relation is then expressed as the symmetry $e_t(\alpha) = e_t(1-\alpha)$, satisfied by the generating function e_t of this random variable. This gives a statistical expression of the second law of thermodynamics: an increase $+s$ (where s is nonnegative) of entropy between 0 and t is more likely by a universal factor e^{+st} than a decrease $-s$: see section 2.3.

It is then a natural question whether, considering a definition of the variation of heat through two-time measurements, one can obtain a statistical formulation of the first law of thermodynamics, and whether this formulation is summarized by a symmetry of the associated generating functions χ_t. Such a symmetry was proposed by Andrieux, Gaspard, Monnai and Tasaki in [5], but the mathematical proof of the symmetry in [5] has a flaw, which we fixed in [P21] by exhibiting a regularity condition on the interactions, that turns out to be necessary for the symmetry to hold. Contrary to the symmetries of e_t, there is no finite-time symmetry for χ_t, and it is necessary to consider the limit as $t \to \infty$. This symmetry and its implications are studied in [P12] regarding the total energy of the system, and in [P21] regarding the detailed heat in each reservoir. We will see that combining the symmetries of functionals χ_+ and e_+ obtained in the limit $t \to \infty$, allows to prove the Kubo formulas, Onsager's reciprocity relations, and a fluctuation-dissipation theorem for fluxes in the steady state, if variations are considered in the sense of two-time measurements. These two types of symmetries could therefore be the universal properties extending the linear response theory beyond "near equilibrium".

Note that, even if definitions of e_t and χ_t can be given directly for infinite systems thanks to the Tomita–Takesaki modular theory, we make the choice of giving definitions for the variations of entropy or heat by two-time measurements on finite systems only, and to consider the "thermodynamic limit" (which makes systems infinite) only at the level of the distributions of these variations. This allows us to avoid involved algebraic considerations which are replaced by elementary probabilistic arguments.

3. Repeated measurements of a system: the outcomes and the system

Let us remark that the measurements considered at the beginning of this chapter are projective measurements. There exists another class of measurements, called *generalized measurements*, that arise among other situations in indirect measurements. To introduce these indirect measurements, let us call $\mathcal{S}$ the considered system and denote $\mathcal{H}_{\mathcal{S}}$ (instead of $\mathcal{H}$) the Hilbert space that describes it. Suppose then that $\mathcal{S}$ is initially in state ρ, and that one carries out the following experiment:

1. one couples the system $\mathcal{S}$ with an "environment" $\mathcal{E}$ described by $\mathcal{H}_{\mathcal{E}}$ initially in the state ξ, and lets them interact according to the unitary U of $\mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{E}}$;
2. one makes a mesurement of an observable M that acts on $\mathcal{H}_{\mathcal{E}}$ only, which we suppose here has a discrete spectrum, so that it can be written $M = \sum_{m \in \mathrm{sp}\, M} m\, \pi_m$;
3. one then disregards the system $\mathcal{H}_{\mathcal{E}}$, in the sense that it will no longer appear in the rest of the experiment (i.e. will no longer interact, or be measured upon).

The result of step 1. is that after the interaction, the union of the two systems is in the state $U(\rho \otimes \xi)U^*$. The Born rule then shows that the observation of outcome m in step 2. has probability equal to $\mathrm{tr}\,\big(\mathrm{Id}_{\mathcal{H}_{\mathcal{S}}} \otimes \pi_m\, U(\rho \otimes \xi) U^*\big)$. One can similarly give an explicit expression for the state of the system after step 2, conditionally on observing outcome m; then, after step 3, one must consider that the state of $\mathcal{H}_{\mathcal{S}}$ is the partial trace[1] along $\mathcal{H}_{\mathcal{E}}$ of the latter conditional state. The derived expressions are cumbersome, but can be simplified: if one denotes

$$\Phi_m(\rho) := \mathrm{tr}_{\mathcal{H}_{\mathcal{E}}}\,\big((\mathrm{Id}_{\mathcal{H}_{\mathcal{S}}} \otimes \pi_m)\, U(\rho \otimes \xi) U^*\, (\mathrm{Id}_{\mathcal{H}_{\mathcal{S}}} \otimes \pi_m)\big) \tag{2}$$

then the result of the experiment corresponding to steps 1,2. and 3. is that the measurement outcome, denoted by m_1, is m with probability

$$\mathbb{P}_\rho(m_1 = m) = \mathrm{tr}\,\big(\Phi_m(\rho)\big), \tag{3}$$

and that conditionally on the measurement outcome m, the state of the system $\mathcal{S}$ becomes

$$\rho_1(m) = \frac{\Phi_m(\rho)}{\mathrm{tr}(\Phi_m(\rho))}. \tag{4}$$

[1] The partial trace along $\mathcal{H}_{\mathcal{E}}$, denoted $\mathrm{tr}_{\mathcal{H}_{\mathcal{E}}}$ is the map $\mathrm{Id} \otimes \mathrm{tr}$ from $\mathcal{B}(\mathcal{H}_{\mathcal{S}}) \otimes \mathcal{B}(\mathcal{H}_{\mathcal{E}})$ to $\mathcal{B}(\mathcal{H}_{\mathcal{S}})$, therefore defined by $\mathrm{tr}_{\mathcal{H}_{\mathcal{E}}}(A \otimes B) = \mathrm{tr}(B)A$

Remark that, in the case where ξ is a pure state $|\Omega\rangle\langle\Omega|$ and that M is non-degenerate, every Φ_m is of the form $\rho \mapsto V_m \rho V_m^*$, and then the evolution preserves the purity of ρ, in the sense that every ρ_n will be pure if ρ is. The non-degeneracy assumption on M is natural, since it means that a measurement outcome specifies a pure state. The purity assumption for ξ becomes natural in situations where one makes, before step 1, a non-degenerate measurement of the state of $\mathcal{H}_\mathcal{E}$, as for example when one considers two-time measurements of M. Remark also that indirect measurements are not the only situations leading to generalized measurements: if for example a projective measurement of M is followed by an evolution described by the unitary U (of $\mathcal{H}_\mathcal{S}$ alone this time) then the effect of these two steps is described by (3) and (4) with $\Phi_m(\rho) = V_m \rho V_m^*$ where $V_m = U\pi_m$.

Contrary to projective measurements, in generalized measurements subsequent measurements are not determined by the first outcome: returning to the case of indirect measurements, if after the first measurement one repeats the experiment corresponding to steps 1. through 3. with the updated system $\mathcal{S}$, but with a "new" environnement $\mathcal{E}_2$, then in general one does not obtain the same outcome for the second measurement as one did for the first. If one iterates n times these generalized measurements with environments $\mathcal{E}_1$,..., $\mathcal{E}_n$, one obtains a n-tuple of outcomes $m_1, \ldots, m_n$ and a state ρ_n which is the updated state conditional on those n outcomes; the stochastic process $(\rho_n)_n$ (or sometimes the process $(x_n, \rho_n)_n$) is called a *quantum trajectory*. The environments $\mathcal{E}_1$, $\mathcal{E}_2$,... are often called *probes* since they serve that purpose in experiments of cavity quantum electrodynamics, such as those of [70, 99, 122]; the latter experiments are the main physical motivation for the present study of repeated indirect measurements. It is then natural to ask what the statistical properties of $(m_n)_n$ are, and what the behaviour of $(\rho_n)_n$ as $n \to \infty$ will be.

To study the behaviour of $(m_n)_n$, we extend slightly the above framework and assume that the parameters of the generalized measurement (for example the parameters ξ, U, M in the indirect measurements) can depend on earlier measurement outcomes. We then describe the measurements not with $m_n \in \operatorname{sp} M$ but with $x_n \in V$, where V is a discrete configuration space, and the maps $(\Phi_m)_{m\in \operatorname{sp} V}$ are replaced by $(\Phi_{i,j})_{i,j\in V}$, where for each j the $(\Phi_{i,j})_{i\in V}$ are the maps that apply during the $n+1$-th measurement if the n-th led to $x_n = j$. The derived process was called *open quantum walk* in [7], and suggested as an analogue of Markov chains that would be relevant to quantum models, with transitions affected by the internal degree of freedom $(\rho_n)_n$ (let us point out immediately that neither $(x_n)_n$, nor $(m_n)_n$ in the preceding case, are in general Markov chains). It is then natural to ask what one can say of the distribution of $(x_n)_n$: one obtains a law of large numbers, a central limit theorem and a large deviations principe from the study of a certain expression for the generating function of $(x_n)_n$; this corresponds to articles [P13, P15]. To push further the analogy with Markov chains, one can investigate the waiting times or the number of visits by $(x_n)_n$ of a configuration i in V and the universality of notions of recurrence for $(x_n)_n$, and this is considered in [P21]. These different topics are described in section 3.1.

Another natural question concerning $(m_n)_n$ is the "appearance of the arrow of time", i.e. the asymmetry in the measurement outcomes. This can be summarized into a practical question: if one is given a list $(m_1, \ldots, m_n)$ which one knows is an actual sequence of outcomes but displayed either in the right order or reversed, can one determine which direction is correct? This is a question of statistical hypothesis testing, and requires the study of the regularity of the Rényi relative entropy of $\mathbb{P}_\rho$ and $\widehat{\mathbb{P}}_\rho$, respectively the distribution of $(m_1, \ldots, m_n)$ and that of the reversed outcomes (which can be $(m_n, \ldots, m_1)$ but one can consider a more elaborate notion of reversal). Answering the question of regularity requires the use of the thermodynamic formalism (see [28]), but the measurements $\mathbb{P}_\rho$ and $\widehat{\mathbb{P}}_\rho$ do not fit into the standard classes of distributions, and we had to apply "non-additive thermodynamic formalism". Our results (described in [P20]) immediately extend to the case where one would like to determine if the list

$(m_1, \ldots, m_n)$ was obtained from indirect measurements carried out with the protocol induced by a set of parameters ξ_1, U_1, M_1, or with that induced by another set ξ_2, U_2, M_2. All these points are discussed in section 3.2.

In section 3.3, we ask what can be said of the long-time behaviour of the process $(\rho_n)_n$. Very few results were known, many of them due to Kümmerer and Massen. One result concerned the fact that under a fairly general condition, not only does the evolution preserve the purity of ρ, but ρ_n is asymptotically pure when $n \to \infty$ (see [95]), regardless of the initial condition. It is therefore natural to restrict the study of $(\rho_n)_n$ to the case of pure states, and then the question concerns a Markov chain on the projective sphere of $\mathcal{H}_\mathcal{S}$, that is formulated in terms of random product of matrices. We show in [P24] that the purification condition of [95] implies a convergence of the distribution of $(\rho_n)_n$ to an invariant distribution, at exponential speed in the first Wasserstein distance.

Last, we finish chapter 3 with sections 3.4 and 3.5 that introduce continuous-time extensions of the results from sections 3.3 and 3.3, both extensions described in [P25] and [P27] respectively. In both cases, we will only give the relevant construction for the model, and indicate how to adapt the proofs from the discrete-time case.

4. Time-dependent systems: the adiabatic case

Landauer's principle (stated in 1961 in [90]) can be summarized as follows: the irreversible transformation of the state of a system $\mathcal{S}$ by interaction with an environment $\mathcal{E}$ initially at thermal equilibrium at temperature T has a minimal energetic cost, and this can be summarized as

$$\beta \Delta Q_\mathcal{E} \geq \Delta S_\mathcal{S} \tag{5}$$

where $\beta = T^{-1}$ is the inverse temperature[2], $\Delta Q_\mathcal{E}$ is the variation of free energy of $\mathcal{E}$ and $\Delta S_\mathcal{S}$ the variation of entropy of $\mathcal{S}$. In addition, one expects inequality (5) to be saturated when the interaction is adiabatic, that is, obtained by an infinitely slow evolution of a full system $\mathcal{S} \vee \mathcal{E}$ initially at equilibrium.

A first satisfactory proof of Landauer's principle was given by Reeb and Wolf in [111]; this proof is written in the quantum formalism, and in finite dimension, in which case one can not observe the adiabatic saturation. The article [79] extended the above approach to the case where the environment is described by a general C*-algebra, allowing to prove the saturation of (5), in the adiabatic limit. The article [P19] studies the same problem modelling the environment by a "repeated interactions system".

A *repeated interactions system* (RIS) is made up of a fixed system $\mathcal{S}$ as above, that interacts during one unit of time with a system $\mathcal{E}_1$, following a Hamiltonian dynamics, before the coupling between $\mathcal{S}$ and $\mathcal{E}_1$ is turned off and $\mathcal{S}$ interacts with a system $\mathcal{E}_2$ during the next unit of time, and so on. This setup is similar to that of e.g. section 3, and we only use the term repeated interactions system to emphasize the role of the sequence $(\mathcal{E}_n)_n$ as a model for an environment. Indeed, in this model the set of all $\mathcal{E}_n$ has by definition two of the characteristics expected from a reservoir: an infinite total energy, and (at least when all systems $\mathcal{E}_n$ are identical) a relaxation time small before that of $\mathcal{S}$. The adiabatic case is described by varying slowly (with variations of order $1/T$ at each step) the parameters of $\mathcal{E}_n$ and of the interaction between $\mathcal{S}$ and $\mathcal{E}_n$. The article [P19] then considers the question of saturation for these adiabatic RIS, with the same definitions of $\Delta S_\mathcal{S}$ and $\Delta Q_\mathcal{E}$ as [111] and [79]. The study of saturation was carried out in [P19]; however, the initial question deserved to be improved.

Indeed, the quantities $\Delta Q_\mathcal{E}$ and $\Delta S_\mathcal{S}$ of [79, 111] are average quantities. One could hope to refine Landauer's principle to a relationship between the distributions of two random variables corresponding

[2] One should actually write $\beta = (k_B T)^{-1}$ where k_B is Boltzmann's constant, but we will fix that constant to 1

to variations, in the sense of two-time measurements, of the entropy of the system $\mathcal{S}$ on the one hand, and the entropy of the chain $\mathcal{E}_1, \mathcal{E}_2, \ldots$ on the other. The generating functions of these random variables can then be expressed from the product of deformations of maps similar to (2) at times n/T, $n = 1, \ldots, T$; to study the asymptotics of these generating functions, we develop two technical results. The first is an "adiabatic theorem" describing the limit as $T \to \infty$ of such a product of maps. Since this limit depends on the peripheral spectra of these deformations, it was necessary to explicit the form of this peripheral spectrum. These two results are decribed in sections 4.2 and 4.3.

We then obtain a statistical formulation of Landauer's principle along the trajectory associated with a repeated interactions system, with saturation under a specific and explicit condition. When this condition is not satisfied, our results are analogous to those of section 3.2 on the Rényi relative entropy between the distributions for forward and backward measurements. All these results are discussed in section 4.4.

5. Some topics that were left aside

Some of my works will not be mentioned in the rest of this document; we discuss them here shortly.

Articles [P6] and [P7] study quantum systems composed of reservoirs $\mathcal{R}_1, \ldots, \mathcal{R}_\ell$, initially at thermal equilibrium at different temperatures, and put into contact. As we wrote in section 1, one expects the global system to settle into a steady state ρ_+, and one can then be interested in the average heat flux $\rho_+(\Phi_i)$ coming out of $\mathcal{R}_i$, and to its dependency on the temperatures of the other reservoirs. This study is carried out here in a C*-algebraic setup, in the case where the dynamics of the system is quasi-free, so that all relevant objects can be expressed in terms of the wave operators for the one-particle evolutions, and that one can obtain explicit formulas for the $\rho_+(\Phi_i)$ without assuming that one is close to equilibrium. These articles are entirely written in the algebraic formalism that we tried to avoid here.

The article [P14] (which contains results that we will use in the rest of this document, but which we will however not detail) gives a general result regarding the decomposition of a quantum channel (the type of operator that describes in discrete time the evolution of an open system, as e.g. 2) into a direct sum of irreducible channels, and describes the full set of invariant states, which we can show is a simplex only under a condition of uniqueness of decompositions of the Hilbert space $\mathcal{H}$ into sums of supports of projectors reducing Φ. The originality of this article lies in this description of invariant states, which had only been given in continuous time and with a lack or rigour that could cast a doubt on its exactness. The latter precisions on the form of invariant states are necessary to obtain the general form of invariant measures for the stochastic evolution of quantum trajectories, as discussed in article [P24], and described in section 3.3.

The article [P16] concerns a refinement of the quantum Stein's lemma. Assume that one has a sequence $\mathcal{O}_n = \mathcal{B}(\mathcal{H}_n)$ of algebras, which have "true" states given either by the sequence $(\rho_n)_n$, or by the sequence $(\sigma_n)_n$, and that one wishes to determine which is the correct sequence, based on the measurement of test observables $(T_n)_n$, where $T_n \in \mathcal{O}_n$. Then, for all τ strictly smaller than $S(\rho|\sigma) := \lim_{n\to\infty} \frac{1}{n} S(\rho_n|\sigma_n)$, one can find tests $(T_n)_n$ (where $T_n \in \mathcal{O}_n$) such that the type 2 error decreases as $\mathrm{e}^{-n\tau}$, and the type 1 error is arbitrarily small. If on the other hand one wants the type 2 error to decrease like $\mathrm{e}^{-n\tau}$ with τ strictly larger than $S(\rho|\sigma)$, then necessarily the type 1 error will tend to 1 (this is the "strong converse"). The article [P16] refines this dependency by showing that the type 1 error optimal under the condition that the type 2 error decreases like $\mathrm{e}^{-nS(\rho|\sigma)-\sqrt{n}\,v}$ is proportional to $\mathbb{P}(Z \le v)$ where Z follows a standard normal distribution – all of this under regularity conditions on the extension to complex parameters of the map that to α associates the α-relative Rényi entropy of the two states. Suprisingly enough, we haven't found any mention of this problem in the literature on classical hypothesis testing.

Last, the article [P18] considers a quantum dynamical semigroup $(\mathrm{e}^{t\mathcal{L}})_{t\in\mathbb{R}_+}$ on a finite-dimensional space, which is viewed in the literature as generating a "quantum diffusion". This diffusion semigroup has the particularity that it preserves the Gaussian character (see [76]) of states. Contrary to the situations studied in other articles, we do not have here convergence of $\rho_t := \mathrm{e}^{t\mathcal{L}}(\rho)$. We study the rate of increase to infinity of the entropy of ρ_t by functional inequalities that allow to show the rate of decrease of the largest eigenvalue of ρ_t. This article was an opportunity to work with functional inequalities in the quantum framework.

Chapter 1

Fluctuation-dissipation and non-commutative central limit

This chapter discusses the results obtained in the articles [P8, P9, P10]. These results constitute a first attemps to give a meaning to the joint laws of fluctuations of observables in a quantum system. These are the only results, in this thesis, that belong to the domain of "non-commutative probability" or "quantum probability". The context that motivated these studies is the linear response theory for out-of-equilibrium open quantum systems, as studied in the articles [P6, P7]. To detail the content of the latter articles would require an introduction to the theory of dynamical systems on C*-algebras, which would take us too far from the main topic of the present document. We will therefore content ourselves with the bare minimum required to motivate the question of joint laws of fluctuations.

The quantum statistical mechanics of out-of-equilibrium systems, as practiced in the articles [P6, P7] (and before that in e.g. [78, 115], which will be our main recommended references for this paragraph), consider a system made of infinite reservoirs $\mathcal{R}_1,\ldots,\mathcal{R}_\ell$, which are initially in states of thermal equilibrium at parameters $\beta_1,\ldots,\beta_\ell$, and that are coupled to one another from time $t=0$, through local interactions and possibly through a "small" system $\mathcal{S}$. If $\ell=1$, or equivalently if $\beta_1=\ldots=\beta_\ell=\beta$, then one expects that in the limit $t\to\infty$, the full system $\mathcal{S}\vee\mathcal{R}_1\vee\ldots\vee\mathcal{R}_\ell$ settles into a state of thermal equilibrium at temperature β. If instead $\ell>1$ and the β_i are not identical – this is the "out-of-equilibrium" case – then at best, the system will settle as $t\to\infty$ into a steady state ρ_+ (ofen called NESS for "non-equilibrium steady state"), and one expects to observe transport, with heat fluxes "out of $\mathcal{R}_i$", represented by observables Φ_i. However, studying the properties of ρ_+ and of the fluxes Φ_i in the general case is difficult. A first approach of the out-of-equilibrium situation uses the fact that, under fairly weak conditions, one can show that if $\beta_1=\ldots=\beta_\ell$, then $\rho_+(\Phi_i)=0$ for all i. Under some regularity assumptions, one should be able to prove that if $\sup_i|\beta_i-\beta_{\mathrm{eq}}|<\epsilon$ for a certain reference value β_{eq} of β, then

$$\rho_+(\Phi_i)=\sum_{j=1}^{\ell}L_{i,j}(\beta_j-\beta_{\mathrm{eq}})+o(\epsilon). \tag{1}$$

The centre of attention is then shifted to the coefficients $L_{i,j}$. The study of these coefficients is the basis of linear response theory (which goes back to Onsager in the thirties, see chapter 4 of [86]). In the classical (i.e. not quantum) setting, linear response theory has three pillars. To formulate them, let us ignore for an instant the distinction between classical and quantum settings, and assume that on an algebra of observables $\mathcal{O}$ (which would actually have to describe an infinitely extended system for the results mentioned below to hold), we have a group of automorphisms $(\tau^t)_t$. The three pillars, that suppose a property of time-reversal invariance (which we will make precise in due time) are then:

- the Kubo formula

$$L_{i,j} = \frac{1}{2}\int_{-\infty}^{+\infty} \rho_{\text{eq}}\Big(\big(\Phi_i - \rho_{\text{eq}}(\Phi_i)\big)\tau^t\big(\Phi_j - \rho_{\text{eq}}(\Phi_j)\big)\Big)\,\mathrm{d}t \tag{2}$$

  where ρ_{eq} is the NESS in the case $\beta_1 = \ldots = \beta_\ell$;

- Onsager's reciprocity relations

$$L_{i,j} = L_{j,i} \text{ for all } i, j\,; \tag{3}$$

- the fluctuation-dissipation relation, which says that the spontaneous time-fluctuations of the Φ_i in the equilibrium state ρ_{eq} are centered Gaussian, with covariance matrix $(2L_{i,j})_{i,j}$.

Proving (1), the Kubo formula and the Onsager relations (as was done in articles [P6,P7] for some classes of quantum models) requires a definition of the dynamical system $(\mathcal{O}, \tau^t)$ but does not raise any conceptual problem. On the contrary, formulating the fluctuation-dissipation relation in a quantum setup requires a clarification of the meaning of fluctuations. Consider for now without discussion that the fluctuations of Φ_i on the time interval $[0, t]$ are represented by the operator

$$\tilde{\Phi}_{i,t} = \frac{1}{\sqrt{t}}\int_0^t \tau^s\big(\Phi_i - \rho_+(\Phi_i)\big)\,\mathrm{d}s.$$

The classical statement of the fluctuation-dissipation theorem concerns *joint* fluctuations, and more precisely the limiting joint law of $\tilde{\Phi}_{i,t}$ and $\tilde{\Phi}_{j,t}$ when $t \to \infty$. Unfortunately, in the quantum case, there is no satisfactory definition for this joint law. This is what we will explain now.

As we wrote on page 20, von Neumann's spectral theorem and bounded Borel functional calculus for self-adjoint operators allow to define the distribution of an observable X in the state ρ: if $X = \int x\,\mathrm{d}\xi_X(x)$ then this distribution is $\mu_X^\rho := \rho \circ \xi_X$, that satisfies $\rho\big(f(X)\big) = \int f(x)\,\mathrm{d}\mu_X^\rho(x)$. So far one remains in the domain of classical probability theory and if, for example, one has a sequence of observables $(X_n)_n$ and a sequence of states $(\rho_n)_n$, one can discuss the asymptotics (in the sense of narrow convergence) of the distribution $\mu_{X_n}^{\rho_n}$ of X_n in state ρ_n. Similarly, if two operators X and Y commute, then they can be written as integrals of the same spectral measure[1] ξ sur $\mathbb{R}^2$:

$$X = \int_{\mathbb{R}^2} x\,\mathrm{d}\xi_{X,Y}(x, y) \qquad Y = \int_{\mathbb{R}^2} y\,\mathrm{d}\xi_{X,Y}(x, y)$$

and one can therefore define $\mu_{X,Y}^\rho := \rho \circ \xi_{X,Y}$ on $\mathbb{R}^2$ satisfying

$$\rho\big(f(X)g(Y)\big) = \int f(x)g(y)\,\mathrm{d}\mu_{X,Y}^\rho(x, y) \tag{4}$$

for all bounded continuous f and g, which defines a joint law of X and Y in the state ρ. The non-commutativity enters the picture as soon as one tries to discuss the joint law of non-commuting X and Y, which, the folklore says, does not exist. One precise statement among many is the following: there does not exist a map $\rho \mapsto \mu_{X,Y}^\rho$ from the set of positive linear forms on $\mathcal{B}(\mathcal{H})$, to the set of probability measures on $\mathbb{R}^2$, such that for all ρ relation (4) holds and

$$\rho\big(f(X)\big) = \int_{\mathbb{R}^2} f(x)\,\mathrm{d}\mu_{X,Y}^\rho(x, y) \qquad \rho\big(g(Y)\big) = \int_{\mathbb{R}^2} g(y)\,\mathrm{d}\mu_{X,Y}^\rho(x, y) \tag{5}$$

[1] Because the von Neumann algebra generated by X and Y is commutative, it admits a functional calculus for Borel functions, and that suffices to construct $\xi_{X,Y}$ from indicator functions.

for all bounded continuous f and g (this[2] can be proven using Theorem 3.2.1 of [44]). In particular, relation (4) cannot be true for all ρ.

To return to our attempt to prove a quantum fluctuation-dissipation theorem, we see that the trouble begins already when one tries to formulate it: since operators $\tilde{\Phi}_{i,t}$ and $\tilde{\Phi}_{j,t}$ do not commute in general, one can not consider their joint law and a fortiori one can not discuss the limit of that law.

1. Convergence of pseudo-characteristic functions and consequences

To anyway give a meaning to the joint behaviour of $\tilde{\Phi}_{i,t}$ and $\tilde{\Phi}_{j,t}$, we return to a more operational approach: if $(X_n)_n$ and X are observables and $(\rho_n)_n$ and ρ are states, to say that the distribution of X_n in the state ρ_n converges to μ_X^ρ means that, for a large class of Borel sets E, it is enough to consider large n for the event "observing whether X_n takes a value in E when the system is in state ρ_n" to have a success probability close to that of "observing whether X takes a value in E when the system is in state ρ". In other words„ the observables $(X_n)_n$ and states $(\rho_n)_n$ define an experiment that approximates the experiment corresponding to observable X and state ρ.

According to the canon of quantum mechanics, the lack of a joint law in a state ρ for two observables X and Y that do not commute is the expression of the impossibility to measure these two observables simultaneously. It is nevertheless possible to measure first X, and then Y. The Born rules mentioned on page 20 show that the probability to observe X in E, then Y in F (where E, F are two Borel sets) is

$$\rho\big(\mathbb{1}_E(X)\mathbb{1}_F(Y)\mathbb{1}_E(X)\big). \tag{6}$$

This quantity, as a function of $E \times F$, does not in general define a measure on $\mathbb{R}^2$ (it is not additive in E[3]); however, (6) does associate a probability to a certain experiment. If one considers for example sequences $(\rho_n)_n$ of states, and sequences $(X_n)_n$, $(Y_n)_n$ of observables, then one can hope that $\lim_n \rho_n\big(\mathbb{1}_E(X_n)\mathbb{1}_F(Y_n)\mathbb{1}_E(X_n)\big)$ converges to a quantity of the form $\rho\big(\mathbb{1}_E(X)\mathbb{1}_F(Y)\mathbb{1}_E(X)\big)$, in which case one can again view $(X_n, Y_n)_n$ and $(\rho_n)_n$ as defining an experiment that approximates the experiment given by (X, Y) and ρ.

Since quantities like (6) are hard to manipulate, it is natural to wonder if it can suffice to consider them in the case where functions $\mathbb{1}_E$, $\mathbb{1}_F$ are replaced by other, more practical functions. In the "classical" case where one considers a unique sequence of variables X_n, the Lévy–Cramér continuity theorem tells us that if

$$\rho_n\big(\mathrm{e}^{\mathrm{i}\alpha X_n}\big) \underset{n\to\infty}{\longrightarrow} \rho\big(\mathrm{e}^{\mathrm{i}\alpha X}\big)$$

for all $\alpha \in \mathbb{R}$, then $\rho_n\big(f(X_n)\big) \underset{n\to\infty}{\longrightarrow} \rho\big(f(X)\big)$ for all bounded continuous function f. One can then extend this convergence to any bounded Borel function for which the set of discontinuity points has zero measure under probability μ_X^ρ, and therefore to some indicator functions (see for example Theorem 29.2 in [22]). The article [P9] investigates a result of this type, starting from *pseudo-characteristic functions* that in the present situation would be all functionals obtained by taking the average in state ρ_n of products e.g. of operators $\mathrm{e}^{\mathrm{i}\alpha_1 X_n}$ and $\mathrm{e}^{\mathrm{i}\alpha_2 Y_n}$.

To give a rigorous statement of our results, assume that we have for all $t \in \mathbb{R} \cup \{\ \}$:

[2] Surprisinly, the literature only rarely gives a precise statement for the impossibility to define a joint law, and even sometimes claims that such a joint law cannot exist even for ρ fixé (which is absurd), see [106] for a discussion of this point.

[3] This is why we do not talk about "measuring X verifying if the result is in E", but about "measuring if X is in E"

1. a von Neumann algebra $\mathfrak{M}_t$ acting on a Hilbert space $\mathcal{H}_t$;

2. a state ρ_t on $\mathfrak{M}_t$;

3. a family $\tilde{A}_t^{(1)}, \ldots, \tilde{A}_t^{(p)}$ of self-adjoint operators on $\mathcal{H}_t$, affiliated to $\mathfrak{M}_t$.

(The addition of the void index "$t =$ " simply means that we also have a von Neumann algebra $\mathfrak{M}$ acting on $\mathcal{H}$ and equipped with a state ρ, etc.).

The reader who does not know the meaning of the above terms can safely replace points 1,2,3. by the following, at the cost of accepting a little flexibility when applying these results in section 3:

1. an algebra $\mathcal{B}(\mathcal{H}_t)$;

2. a density matrix $\rho_t \in \mathcal{S}(\mathcal{H}_t)$;

3. a family of a priori unbounded self-adjoint operators $\tilde{A}_t^{(1)}, \ldots, \tilde{A}_t^{(p)}$.

Let us define what will be a recurring assumption in the rest of this section:

(CPF) For all m in $\mathbb{N}^*$, all $\alpha_1, \ldots, \alpha_m$ in $\mathbb{R}$, all $j_1, \ldots, j_m$ in $\{1, \ldots, p\}$, one has

$$\lim_{t\to\infty} \rho_t\big(\mathrm{e}^{\mathrm{i}\alpha_1 \tilde{A}_t^{(j_1)}} \ldots \mathrm{e}^{\mathrm{i}\alpha_m \tilde{A}_t^{(j_m)}}\big) = \rho\big(\mathrm{e}^{\mathrm{i}\alpha_1 \tilde{A}^{(j_1)}} \ldots \mathrm{e}^{\mathrm{i}\alpha_m \tilde{A}^{(j_m)}}\big).$$

Assumption **(CPF)** therefore means that any average, in the state ρ_t, of products of operators $\mathrm{e}^{\alpha_k \tilde{A}_t^{(k)}}$ (permutations and repetitions being allowed) converges as $t \to \infty$ to the average in the state ρ of the same product of operators $\mathrm{e}^{\alpha_k \tilde{A}^{(k)}}$.

We then have the following theorem, where for f a Borel function, one denotes $\mathcal{D}(f)$ the set of its discontinuity points.

Theorem 1.1 ([P9]). *Under assumption* **(CPF)**, *one has for all $m \in \mathbb{N}^*$ and all bounded continuous functions $f_1, \ldots, f_m$*

$$\lim_{t\to\infty} \rho_t\big(f_1(\tilde{A}_t^{(1)}) \cdots f_m(\tilde{A}_t^{(m)})\big) = \rho\big(f_1(\tilde{A}^{(1)}) \cdots f_m(\tilde{A}^{(m)})\big), . \tag{7}$$

In addition, one can extend the convergence (7) to a class of discontinuous functions: there exist families $S_1, \ldots, S_m$ of probability measures such that (7) holds for any bounded Borel functions $f_1, \ldots, f_m$ as soon as for all $j = 1, \ldots, m$ one has $\mu_j\big(\mathcal{D}(f_j)\big) = 0$ for all $\mu_j \in S_j$.

Remark 1.2. If the variables $\tilde{A}_t^{(1)}, \ldots, \tilde{A}_t^{(p)}$ commuted, it would be enough to choose $S_j = \{\mu^{\rho}_{\tilde{A}^{(j)}}\}$ for all j. In general, one cannot choose the S_j in a unique manner, and one cannot choose them independently for different j (but acceptable and semi-explicit choices are given by the proof), and this is illustrated by various examples in section 4 of [P9]. The proof and those examples show that the reason for the above impossibilities is that, for example, the support of the distribution of $\mathrm{e}^{+\mathrm{i}\alpha_2 \tilde{A}^{(2)}} f_1(\tilde{A}^{(1)}) \mathrm{e}^{-\mathrm{i}\alpha_2 \tilde{A}^{(2)}}$ in the state ρ depends in general on α_2: the unitary $\mathrm{e}^{+\mathrm{i}\alpha_2 \tilde{A}^{(2)}}$ can "make visible by ρ a spectral subspace of $\tilde{A}^{(1)}$ that was invisible when $\alpha_2 = 0$".

The appearance of these families S_j makes it in general hard to know which discontinuities are allowed for the functions f_j; luckily the situation is simpler as soon as the state ρ is faithful:

PROPOSITION 1.3 ([P9]). *If ρ is faithful, then under assumption* **(CPF)**, *one has* (7) *for any bounded Borel functions $f_1, \ldots, f_m$such that $\mu^{\rho}_{\tilde{A}^{(j)}}\big(\mathcal{D}(f_j)\big) = 0$ for all $j = 1, \ldots, m$.*

Remark 1.4. The proof of Theorem 1.1 is essentially based on repeated applications of the classical Lévy–Cramér continuity theorem after a Fourier transform, and on a careful use of the Cauchy–Schwarz inequality. We will not say more about this technical proof. Proposition 1.3 comes from the fact that, if a state ρ is faithful, then the distributions μ^{ρ}_{A} and $\mu^{\rho}_{UAU^*}$ are mutually absolutely continuous when U is unitary. Remark also that, as in the classical case, one can extend this result to unbounded functions satisfying some domination assumptions under the measures that are elements of the sets S_j. Again we will not state these assumptions, even though they are used in the extension of [88] that we discuss in remark 1.9 below.

Remark 1.5. The convergence of pseudo-characteristic has been extensively studied (see [1, 2, 66, 67, 97]), but it seems that most authors viewed this convergence as a sufficient indication of the relevance of the "limiting structure" described in our notation by observables $\tilde{A}^{(1)}, \ldots, \tilde{A}^{(p)}$ and by the state ρ of $\mathfrak{M}$. The only cases where the operational implications of that convergence has been studied are, to my knowledge, the article by Cushen and Hudson [39] which considers only pairs P_t, Q_t of operators satisfying a canonical commutation relation $[P_t, Q_t] = \mathrm{i}\,\mathrm{Id}$, so that our pseudo-chlaracteristic functions reduce by the Weyl commutation relation to a Wigner function; and Kuperberg's article [88] which assumes the traciality of the state ρ (that is $\rho(AB) = \rho(BA)$ for all A, B) and uses that assumption when going from pseudo-characteristic to general functions. Remark also that Kuperberg underlines in the last page of [88] the lack of an "analytic theory of non-commutative characteristic functions" which is precisely what we developed ici.

To return to the initial motivation described at the beginning of this chapter: it would suffice, to obtain a satisfactory convergence of sequential measurements of the $\tilde{\Phi}_{1,t}, \ldots, \tilde{\Phi}_{\ell,t}$, to show the convergence of their pseudo-characteristic functions.

2. Central limit theorem – quantum or classical

After proving in the preceding section that one could obtain a convergence result starting from pseudo-characteristic functions, we will now show that, when the $\tilde{A}^{(i)}_t$ are of the form

$$\tilde{A}^{(i)}_t = \frac{1}{\sqrt{t}} \int_0^t \tau^s\big(A^{(i)} - \rho(A^{(i)})\big)\,\mathrm{d}s \quad \text{or} \quad \tilde{A}^{(i)}_t = \frac{1}{\sqrt{t}} \sum_{s=0}^{t} \tau^s\big(A^{(i)} - \rho(A^{(i)})\big)$$

(depending on whether one works in continuous or in discrete time), where each $A^{(i)}$ is an observable in $\mathcal{B}(\mathcal{H})$, $(\tau^t)_t$ is an automorphism group of $\mathcal{O} = \mathcal{B}(\mathcal{H})$ and ρ a τ-invariant state, then under ergodicity assumptions on τ, the convergence of pseudo-characteristic functions of $\tilde{A}^{(1)}, \ldots, \tilde{A}^{(p)}$ associated with states $\rho_t \equiv \rho$ reduces to the convergence of characteristic functions $\rho\big(\mathrm{e}^{\mathrm{i}\alpha \tilde{B}_t}\big)$ for $\tilde{B}_t$ associated with any linear combination B of the $A^{(1)}, \ldots, A^{(p)}$.

Let us begin with a few definitions, assuming as above that one has a τ-invariant state ρ. We will say that a self-adjoint vector space $\mathcal{A}$ is *CLT-admissible* if for all A, B in $\mathcal{A}$, $t \mapsto \big|\rho\big(\big(A - \rho(A)\big)\tau^t\big(B - \rho(B)\big)\big)\big|$ is integrable. If this condition is satisfied, we let

$$L(A, B) = \frac{1}{2} \int_{-\infty}^{+\infty} \rho\Big(\big(A - \rho(A)\big)\, \tau^t\big(B - \rho(B)\big)\Big)\,\mathrm{d}t. \tag{8}$$

Let us remark immediately that if $t \mapsto \left|\rho\big(\big(A-\rho(A)\big)\,\tau^t\big(B-\rho(B)\big)\big)\right|$ is integrable, then

$$\rho\big(\tilde A_t \tilde B_t\big) \underset{t\to\infty}{\longrightarrow} 2L(A,B),$$

so that it is clear that the bilinear function L plays a role in the limiting structure. This structure will be described by a Weyl algebra: remark that

$$\iota(A,B) := \frac{1}{2\mathrm{i}} \int_{-\infty}^{\infty} \omega\big([\tau^t(A),B]\big)\,\mathrm{d}t = 2\,\mathrm{Im}\,L(A,B)\big) \tag{9}$$

defines a bilinear form which is symplectic (i.e. $\iota(A,B) = -\iota(B,A)$ for all A and B). We then denote by $\mathcal{W}(\mathcal{A})$ the C*-algebra generated by the unitaries $W(A)$, indexed by A any self-adjoint element $\mathcal{A}$, that satisfy

$$W(A)\,W(B) = \mathrm{e}^{-\mathrm{i}\iota(A,B)/2} W(A+B), \tag{10}$$

(see [109] for more information on this C*-algebra called "CCR algebra"). Remark that (by Stone's theorem) one can define uniquely self-adjoint operators $\varphi(A)$ by the relation $W(\alpha A) = \mathrm{e}^{\mathrm{i}\alpha\varphi(A)}$. If for $n \in \mathbb{N}^*$ one denotes by $\mathcal{P}_n$ the set of permutations π of $\{1,\ldots,2n\}$ such that for $j=1,\ldots,n$ one has $\pi(2j-1) < \min\big(\pi(2j),\pi(2j+1)\big)$, then the relations

$$\omega_L\big(\varphi(A)\varphi(B)\big) = 2L(A,B),$$

and

$$\omega_L\big(\varphi(A_1)\cdots\varphi(A_n)\big) = \begin{cases} \displaystyle\sum_{\pi\in\mathcal{P}_{n/2}} \prod_{j=1}^{n/2} \omega_L\big(\varphi(A_{\pi(2j-1)}),\varphi(A_{\pi(2j)})\big) & \text{if } n \text{ is even;} \\ 0 & \text{if } n \text{ is odd.} \end{cases} \tag{11}$$

define a state ω_L on $\mathcal{W}(\mathcal{A})$ which is called *quasi-free* (and verifies $\omega_L\big(W(A)\big) = \mathrm{e}^{-L(A,A)}$). The algebra $\mathcal{W}(\mathcal{A})$, equipped with the state ω_L is called the fluctuation algebra by [66, 67, 97] in settings described below.

Remark 1.6. In the case where $L(A,B) \in \mathbb{R}$, one has $\iota(A,B) = 0$. As a consequence, if $L(A,B)$ is real for all A, B in $\mathcal{A}$, the relation (10) shows that the algebra $\mathcal{W}(\mathcal{A})$ is commutative. Operators $\varphi(A^{(1)})$, …,$\varphi(A^{(p)})$, where the $A^{(j)}$ belong to $\mathcal{A}$, then admit a joint distribution, and one shows easily from relations (2) and (11) that this distribution is Gaussian with covariance matrix $\big(L(A^{(i)},A^{(j)})\big)_{i,j}$.

The main result of this section, which is proven in [P8, P10], is the following. Before we state it, let us say that the dynamical system is called ergodic if for all A, B, C in $\mathcal{O}$ one has

$$\frac{1}{t}\int_0^t \rho\big(A\,\tau^s(B)\,C\big)\,\mathrm{d}s \underset{t\to\infty}{\longrightarrow} \rho(AC)\rho(B)$$

(this definition extends that of classical dynamical systems), and that a subset $\mathcal{A}$ of $\mathcal{O}$ is L^1-asymptotically Abelian if for all A, B in $\mathcal{A}$, $t \mapsto \|[A,\tau^t(B)]\|$ is integrable. Ergodicity is typically hard to prove, but there are known critera for a choice of standard models. The L^1-asymptotic Abelianness is also hard to prove (and actually implies the existence of a form of wave operators, which makes it in practice an ergodic property) but the property is known for fermionic systems (see [4, 24, 25]). We will not define the last required property, which is that of modularity of a state ρ, and will be content with writing that it holds for the states that will interest us in the application in (14) and in Theorem 1.11).

Theorem 1.7. *If* $(\mathcal{O}, \rho, \tau)$ *is ergodic,* ρ *is modular, and* $\mathcal{A}$ *is a self-adjoint, CLT-admissible, and* L^1*-asymptotically Abelian vector space such that for all* $A \in \mathcal{A}$*,*

$$\lim_{t\to\infty} \rho(\mathrm{e}^{\mathrm{i}\tilde{A}^{(t)}}) = \omega_L\big(W(A)\big), \tag{12}$$

then for all $m \in \mathbb{N}^*$*, all* $\alpha_1, \dots, \alpha_m$ *de* $\mathbb{R}$*, all* $A^{(1)}, \dots, A^{(m)}$ *of* $\mathcal{A}$*,*

$$\lim_{t\to\infty} \rho(\mathrm{e}^{\mathrm{i}\alpha_1 \tilde{A}_t^{(1)}} \dots \mathrm{e}^{\mathrm{i}\alpha_m \tilde{A}_t^{(m)}}) = \omega_L\big(\mathrm{e}^{\mathrm{i}\alpha_1 \varphi(A^{(1)})} \dots \mathrm{e}^{\mathrm{i}\alpha_m \varphi(A^{(m)})}\big). \tag{13}$$

Remark 1.8. This theorem shows that under its assumptions, it is enough to prove a "classical" central limit property to obtain a convergence of pseudo-characteristic functions, which is associated with a Weyl algebra (and therefore of bosonic nature). Remark that, although we have emphasized the classical nature of property (12), the quantum structure is still present in it, as it is involved in the definition of the functional $\rho(\mathrm{e}^{\mathrm{i}\tilde{A}^{(t)}})$.

Theorems 1.1 and 1.7 have a simple application to the fluctuations in space (and not in time) of a spin system. To define this framework, suppose that $\mathfrak{N}$ is a von Neumann algebra equipped with a normal state η (one can simply think of the case $\mathfrak{N} = \mathcal{B}(\mathcal{H})$ with η a state given by a density matrix). One then considers the product von Neumann algebra

$$\mathfrak{M} = \bigotimes_{t\in\mathbb{N}^*}^{(\eta)} \mathfrak{N}$$

(which is the infinite product stabilized by η, i.e. constructed by considering the GNS representation of $\mathfrak{N}$ associated with η, and taking its product with respect to the associated pure state, see section 2.7.2 of [29]), which is equipped with the product state denoted by ρ. One then identifies $A \in \mathfrak{N}$ with $A \otimes \mathrm{Id} \otimes \ldots \in \mathfrak{M}$ and denotes by τ the right shift on $\mathfrak{M}$. One defines as in the beginning of this section

$$\tilde{A}^{(t)} = \frac{1}{\sqrt{t}} \sum_{s=1}^{t} \big(\tau^s(A) - \eta(A)\big)$$

for $t \in \mathbb{N}^*$. One then has, with the above notation,

$$L(A, B) = \eta(AB) - \eta(A)\eta(B) \qquad \iota(A, B) = \eta\big([A, B]\big).$$

The $\tau^s(A)$ constitute a commutative family, and therefore have a joint distribution under ρ, which is a product law. The classical central limit theorem then implies that (12) holds. The main result of [P10] is obtained by a direct application of Theorems 1.1 and 1.7: for any bounded Borel functions $f_1, \dots, f_p$ and any $A^{(1)}$,…,$A^{(p)}$, one has

$$\lim_{t\to\infty} \rho\big(f_1(\tilde{A}_1^{(t)}) \dots f_p(\tilde{A}_p^{(t)})\big) = \omega_L\Big(f_1\big(\varphi(A^{(1)})\big) \dots f_m\big(\varphi(A^{(m)})\big)\Big), \tag{14}$$

if for all $j \in \{1, \dots, m\}$ the set $\mathcal{D}(f_j)$ of discontinuity points of f_j has zero Lebesgue measure, and if one assumes in addition when $\eta(A_j^2) - \eta(A_j)^2 = 0$ that f_j is continuous at zero.

Remark 1.9. This result was already known under the assumption that the f_j are bounded continuous functions (see [66]). Our contribution is essentially the extension to discontinuous functions thanks to [P9]. One can in addition extend (14) to obtain the convergence of any self-adjoint polynomial in the $\tilde{A}_j^{(t)}$. Remark that Kuperberg proves in [88] that any self-adjoint polynomial in the $\tilde{A}_j^{(t)}$ converges in distribution towards its analogue in the components of a Gaussian vector with covariance matrix $\big(L(A^{(i)}, A^{(j)})\big)_{i,j}$, under the assumption that η is tracial (in which case $L(A, B) \in \mathbb{R}$ for all A,B), and conjectures (see section 4 of [88]) that the traciality assumption is superfluous. Our result does not quite prove the conjecture as polynomials in Gaussian variables are not in general determined by their moments (see [73]), so that a characterization by polynomials does not suffice to prove convergence in distribution.

We will now apply the results of sections 1 and 2 to a physical system.

3. Central limit theorem for fermionic systems

In this section, we will work on a C*-algèbre $\mathcal{O} = \mathrm{CAR}(\mathfrak{h})$, that is, a C*-algebra generated by operators $\psi(f)$, $f \in \mathfrak{h}$, satisfying the so-called canonical anticommutation relations[4] :

$$\psi(f)\psi(g) + \psi(g)\psi(f) = \mathrm{Re}\langle f, g\rangle \,\mathrm{Id}.$$

Note that all operators $\psi(f)$ are bounded, so that the reader can without much harm imagine that they are working with an algebra $\mathcal{B}(\mathcal{H})$ (that would be related with the Araki-Wyss representation of the CAR algebra, see [30]). Examples of such $\mathfrak{h}$ for which the assumptions stated below can be checked are given at the end of section 1.2 of [P8].

We will denote by $\mathcal{O}_e$ the C*-algebra with unit generated by the operators of the form $\psi(f)\psi(g)$, $f, g \in \mathfrak{h}$. Remark that for reasons of gauge invariance, the physical observables of the system must belong to $\mathcal{O}_e$. For $\mathfrak{g}$ a subspace of $\mathfrak{h}$, we denote $\mathcal{A}(\mathfrak{g})$, $\mathcal{A}_e(\mathfrak{g})$ the $*$-algebras (and not C*-algebra) with unit generated respectively by the $\psi(f)$ and by the $\psi(f)\psi(g)$. If $\mathfrak{g}$ is dense in $\mathfrak{h}$, then $\mathcal{A}(\mathfrak{g})$ (respectively $\mathcal{A}_e(\mathfrak{g})$) is dense in norm in $\mathcal{O}$ (respectively $\mathcal{O}_e$).

We fix from now on a self-adjoint operator (not necessarily bounded) h_0 on $\mathfrak{h}$ and one defines an automorphism group $(\tau_0^t)_t$ of $\mathcal{O}$ by $\tau_0^t\big(\psi(f)\big) = \psi(\mathrm{e}^{\mathrm{i}th_0} f)$; an automorphism of this type is called a *Bogoliubov automorphism*. This τ_0 will represent for us the free, i.e. uncoupled, dynamics. To define the coupled dynamics, we denote by δ_0 the generator of τ_0, so that $\tau_0^t = \mathrm{e}^{t\delta_0}$ (see section 3.2.4 of [29]). Let then $V \in \mathcal{A}_e(\mathfrak{g})$, which can therefore be written

$$V = \sum_{k=1}^{K_V} \prod_{j=1}^{n_k} \psi(f_{k_j})\psi(g_{k_j}). \tag{15}$$

We denote $\bar{n}_V := \max(n_1, \ldots, n_{K_V})$ and $\mathcal{F}(V) = \{f_{k_j}, g_{k_j},\ j = 1, \ldots, n_k,\ k = 1, \ldots, K_V\}$. We define a dynamics τ_λ for $\lambda \in \mathbb{R}$ with small enough modulus (see below) by

$$\tau_\lambda^t = \mathrm{e}^{t\delta} \quad \text{where} \quad \delta = \delta_0 + \mathrm{i}\lambda[V, \cdot].$$

Conditions ensuring that this defines a strongly continuous dynamics on $\mathcal{O}$ are given below.

The above result can be proven using the techniques of [4,24,25] (see [81] for a proof) when $\bar{n}_V > 1$, and of [P7] when $\bar{n}_V = 1$.

[4] We denote here $\psi(f)$ in place of $\varphi(f)$ the fermionic field operators to avoid any confusion with the bosonic operators φ of the fluctuation algebra.

Proposition 1.10. *Assume that there exists a subspace $\mathfrak{g}$ which is dense in $\mathfrak{h}$ and such that $t \mapsto \langle f, \mathrm{e}^{ith_0} g\rangle$ is integrable for all $f, g \in \mathfrak{g}$. Let then $\ell_V = \int_{\mathbb{R}} \sup_{f,g\in\mathcal{F}(V)} \left|\langle f, \mathrm{e}^{ith_0} g\rangle\right| \mathrm{d}t$ and*

$$\lambda_V := \frac{1}{2\bar{n}_V K_V \ell_V} \frac{(2\bar{n}_V - 2)^{2\bar{n}_V-2}}{(2\bar{n}_V - 1)^{2\bar{n}_V-1}} \quad \text{if } \bar{n}_V > 1, \qquad \lambda_V := \frac{1}{2K_V\ell_V} \quad \text{if } \bar{n}_V = 1. \tag{16}$$

Then for all $A \in \mathcal{A}_e(\mathfrak{g})$ and $B \in \mathcal{A}(\mathfrak{g})$, one has

$$\sup_{|\lambda|\le\lambda_V} \int_{\mathbb{R}} \left\|[\tau_\lambda^t(A), B]\right\| \mathrm{d}t < \infty.$$

In addition, for $|\lambda| \le \lambda_V$, the following limit exists for all $A \in \mathcal{O}$:

$$\gamma_\lambda^+(A) := \lim_{t\to\infty} \tau_0^{-t} \circ \tau_\lambda^t(A)$$

and defines an automorphism γ_λ^+ of $\mathcal{O}$ called a Møller morphism.

One can then (again see [P7] for the case $\bar{n}_V = 1$ and [81] in the case $\bar{n}_V > 1$), show under an assumption of time-reversal invariance the Kubo formulas and Onsager's reciprocity relations. To discuss our central limit theorem, we need to define our initial state. Let then T be a self-adjoint operator on $\mathfrak{h}$ satisfying $0 \le T \le I$ and $[T, \mathrm{e}^{ith_0}] = 0$ for all t. Let ρ_0 be the quasi-free state on $\mathcal{O}$ with density T, that is, if we denote by $\epsilon(\pi)$ the signature of a permutation $\pi \in \mathcal{P}_n$ (see page 34), then ρ_0 is the unique state such that

$$\rho_0\big(\psi(f_1)\psi(f_2)\big) = \frac{1}{2}\langle f_1, f_2\rangle - \mathrm{i}\ \mathrm{Im}\langle f_1, T f_2\rangle,$$

and

$$\rho_0\big(\psi(f_1)\cdots\psi(f_n)\big) = \begin{cases} \displaystyle\sum_{\pi\in\mathcal{P}_{n/2}} \epsilon(\pi) \prod_{j=1}^{n/2} \rho_0\big(\psi(f_{\pi(2j-1)}), \psi(f_{\pi(2j)})\big) & \text{if } n \text{ is even;} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

We view ρ_0 as the initial state (before any coupling is turned on) of our system. The quantum dynamical system $(\mathcal{O}, \tau_0, \rho_0)$ is ergodic (see example 5.2.21 of [30]); Proposition 1.10 shows that the NESS in the sense of Ruelle (see [115]) $\rho_\lambda^+ := \rho_0 \circ \gamma_\lambda^+$, is τ_λ-invariant and that the system $(\mathcal{O}, \tau_\lambda, \rho_\lambda^+)$ is ergodic as well for $|\lambda| \le \lambda_V$. This state verifies in particular the relation $\lim_{t\to\infty} \rho_0 \circ \tau_\lambda^t = \rho_\lambda^+$. In the sequel, we suppose that $\ker T = \ker(I - T) = \{0\}$, and this assumption ensures that the states ρ_0 and ρ_λ^+ are modular (see Theorem 43.6 of [46]).

We consider from now on the bilinear form L defined by (8) with $\rho = \rho_\lambda^+$ and $\tau^t = \tau_\lambda^t$, and the algebra $\mathcal{W}\big(\mathcal{A}_e(\mathfrak{f})\big)$ associated with L, equipped with a state ω_L and consisting of operators $\varphi(A)$ for $A \in \mathcal{A}_e(\mathfrak{f})$. All these objects are defined in section 2. The main result of [P8] is the following.

Theorem 1.11. *Suppose that there exist a subspace $\mathfrak{g}$ as in Proposition 1.10, and a subspace $\mathfrak{f}$ of $\mathfrak{g}$ such that $t \mapsto \langle f, \mathrm{e}^{ith_0} T g\rangle$ is integrable for all $f, g \in \mathfrak{f}$. Then for all $\lambda \in \mathbb{R}$ with $|\lambda| \le 2^{-8(\bar{n}_V-1)}\lambda_V$, all $m \in \mathbb{N}^*$ and all $A_1, \ldots, A_m \in \mathcal{A}_e(\mathfrak{f})$, one has*

$$\lim_{t\to\infty} \rho_\lambda^+\big(f_1(\tilde{A}_t^{(1)}) \ldots f_m(\tilde{A}_t^{(m)})\big) = \omega_L\Big(f_1\big(\varphi(A^{(1)})\big) \ldots f_m\big(\varphi(A^{(m)})\big)\Big) \tag{17}$$

for any bounded Borel functions $f_1, \ldots, f_p$ such that for all $j \in \{1, \ldots, m\}$ the set $\mathcal{D}(f_j)$ of discontinuity point of f_j has zero Lebesgue measure, if one assumes in addition that f_j is continuous at zero when when $L(A^{(j)}, A^{(j)}) = 0$.

Remark 1.12. If T is of the form $F(h_0)$ with $F \in \mathrm{L}^1$ of integrable Fourier transform, then one can choose $\mathfrak{f} = \mathfrak{g}$. This remark is useful in the application to fermionic systems discussed below.

Let us discuss the proof of Theorem 1.11. From Theorems 1.1, and 1.7, it is enough to prove that for all $A \in \mathcal{A}_e(\mathfrak{f})$ one has

$$\rho_\lambda^+\big(\mathrm{e}^{\mathrm{i}\alpha\tilde{A}_t}\big) \underset{t\to\infty}{\longrightarrow} \mathrm{e}^{-\frac{1}{2}\alpha^2 L(A,A)}$$

for all $\alpha \in \mathbb{R}$. The proof is very different for $\bar{n}_V > 1$ and for $\bar{n}_V = 1$. The former is treated in [P8]; the proof is much too long to be given here, and we will content ourselves below with an indication of the main ideas. The second and much simpler case was announced in [P8] as proved in an later article that was never written, and will serve here as an introduction to the case $\bar{n}_V > 1$.

Let us therefore begin with the case $\bar{n}_V = 1$; the specificity of this case is that $(\tau_\lambda^t)_t$ is still a Bogoliubov dynamics, i.e. there exists h_λ such that $\tau_\lambda^t\big(a(f)\big) = a(\mathrm{e}^{\mathrm{i}th_\lambda} f)$, where h_λ is a finite-rank perturbation of h_0. As a consequence, the Kato-Rosenblum theorem (Theorem XI.8 in [112]) proves the existence of $W^\pm = \lim_{t\to\infty} \mathrm{e}^{+\mathrm{i}th_0}\mathrm{e}^{-\mathrm{i}th_\lambda}$. This implies that ρ_λ^+ is still a quasi-free state with density $T^+ = W_-^* T W_-$. If one considers $A \in \mathcal{A}_e(\mathfrak{f})$, it can be written as a linear combination of products $\varphi(f_1)\ldots\varphi(f_{2\ell})$. If one expands $\tilde{A}_t^p$ in $\sum_{p\in\mathbb{N}} \frac{\mathrm{i}^p\alpha^p}{p!}\rho_\lambda^+(\tilde{A}_t^p)$, one has terms of the form $\rho_\lambda^+(\tilde{A}_t^{(1)}\ldots\tilde{A}_t^{(p)})$, that can be written

$$t^{-p/2}\int_{[0,t]^p} \rho_\lambda^+\Big(\tau_\lambda^{s_1}\big(A^{(1)} - \rho_\lambda^+(A^{(1)})\big)\big)\ldots\tau_\lambda^{s_p}\big(A^{(p)} - \rho_\lambda^+(A^{(p)})\big)\big)\Big)\,\mathrm{d}s_1\ldots\mathrm{d}s_p, \tag{18}$$

where each $\tau_\lambda^{s_k}(A^{(k)})$ is of the form $\varphi(\mathrm{e}^{\mathrm{i}s_k h_\lambda} f_1^{(k)})\ldots\varphi(\mathrm{e}^{\mathrm{i}s_k h_\lambda} f_{2\ell_k}^{(k)})$ (for each p there is an number exponential in p of such terms, which the α^p can absorb). If we now use the quasi-free property of ρ_λ^+, this is non-zero only for p even, and then expands to a sum indexed by pairings $\pi \in \mathcal{P}_{2\ell}$ where $2\ell = 2\ell_1 + \ldots + 2\ell_p$, on points indexed by (j,k) where $k = 1,\ldots,p$ and $j = 1,\ldots,2\ell_k$. If one considers then the pseudograph obtained by collapsing all points associated to the same k, and denotes $c(\pi)$ the number of connected components of this pseudograph, then one can first see that the "centering" caused by the presence of the $\rho_\lambda^+(A^{(k)})$ in (18) translates exactly into the interdiction of those π for which the pseudograph has an isolated point; then, by a change of variable, that the term associated with π in the expansion of (18) can be bounded above in $C^{2\ell}t^{c(\pi)-p/2}$. As a consequence, in the limit $t \to \infty$, only those terms corresponding to π with $c(\pi) = p/2$ survive, and a resummation of these terms shows that

$$\rho_\lambda^+(\tilde{A}_t^p) \underset{t\to\infty}{\longrightarrow} \begin{cases} \dfrac{p!}{(p/2)!}\,L(A,A)^{p/2} & \text{if } p \text{ is even,} \\ 0 & \text{if } p \text{ is odd,} \end{cases} \tag{19}$$

One easily obtains uniform bounds of the form $C^p p!$ for $\rho_\lambda^+(\tilde{A}_t^p)$, which allows to conclude that $\rho_\lambda^+(\mathrm{e}^{\mathrm{i}\alpha\tilde{A}_t})$ converges as $t \to \infty$ to $\mathrm{e}^{-\alpha^2 L(A,A)}$.

Remark 1.13. A similar proof of this case $\bar{n}_V = 1$ is given in a different setting in [47], an article which we have discovered only after [P8] was accepted for publication.

In the case $\bar{n}_V > 1$, one begins by writing

$$\begin{aligned}
&\rho_\lambda^+\Big(\tau_\lambda^{s_1}\big(A^{(1)} - \rho_\lambda^+(A^{(1)})\big)\big)\ldots\tau_\lambda^{s_p}\big(A^{(p)} - \rho_\lambda^+(A^{(p)})\big)\big)\Big) \\
&\qquad = \rho_0\Big(\tau_0^{s_1}\big(\gamma_\lambda^+(A^{(1)}) - \rho_0\circ\gamma_\lambda^+(A^{(1)})\big)\ldots\tau_0^{s_p}\big(\gamma_\lambda^+(A^{(p)}) - \rho_0\circ\gamma_\lambda^+(A^{(p)})\big)\Big)
\end{aligned}$$

and then one can use a Dyson expansion of each $\gamma_+(A^{(k)})$. This expansion can then be written as a series of iterated integrals of commutators. Every commutator can in turn be written (see [81] ou [24]) as a series indexed by trees with a number of vertices equal to the order of the commutator, every term in the series being a monomial in field operators, and the degree of this monomial grows linearly with the order of the commutator. One can then attempt to apply the same technique as before. For each choice of the k pairs of order and tree, one can essentially apply (19), but one no longer has uniform bounds allowing to use a dominated convergence argument. The bulk of the work in [P8] is to show that

$$\int_{[0,t]^p} \left|\rho_0\Big(\tau_0^{s_1}\big(A^{(1)} - \rho_0 \circ (A^{(1)})\big) \dots \tau_0^{s_p}\big(A^{(p)} - \rho_0 \circ (A^{(p)})\big)\Big)\right| \mathrm{d}s_1 \dots \mathrm{d}s_p, \tag{20}$$

where the $A^{(k)}$ are of the form $\varphi(f_1^{(k)}) \dots \varphi(f_{2\ell_k}^{(k)})$ (with the normalization $\|f_m^{(k)}\| \le 1$) has a bound in $C_p\, t^{p/2} D^{\sum \ell_k}$, where D is a constant, the C_p are uniform in $\ell_1, \dots, \ell_p$ and such that $\sum \alpha^p C_p/p!$ has a non-zero radius of convergence. Unfortunately,

- if one bounds (20) directly by an inequality $|\rho_0(A)| \le \|A\|$, the integral implies a bound $t^p D^{\sum \ell_k}$, so that the dependency in t is bad;
- if one expands (20) thanks to the quasi-free property, every term can be bounded as in the case $\bar{n}_V = 1$ as $t^{p/2} D^{\sum \ell_k}$, but the number of terms is of the order $(2\sum \ell_k)!$, so that the dependency in $\sum \ell_k$ is bad.

The idea is therefore to expand (20) using the quasi-free property, but to resum in order to control the growth in the number of terms thanks to inequalities $|\rho_0(A)| \le \|A\|$. We will not say more here, and refer the reader to [P8].

4. Fluctuation-dissipation for fermionic systems

Let us now conclude with the application of the above program to a fluctuation-dissipation theorem. Theorem 1.11 shows that under certain assumptions, the operator algebra most relevant to describe the limit of fluctuations of observables in $\mathcal{A}_e(\mathfrak{f})$) is the Weyl algèbre $\mathcal{W}\big(\mathcal{A}_e(\mathfrak{f})\big)$ on $\mathcal{A}_e(\mathfrak{f})$, for the symplectic form $\iota(A,B) = 2\,\mathrm{Im}\,L(A,B)$. Section 1.2 of [30] gives a list of examples of fermionic open quantum systems for which the assumptions of Theorem 1.11 are verified. One can then wonder in which case the fluctuation algebras $\varphi(A)$ define joint laws.

It is clear from relation (10) defining $\mathcal{W}\big(\mathcal{A}_e(\mathfrak{f})\big)$ that this algebra is commutative if and only if $\iota \equiv 0$ sur $\mathcal{A}_e(\mathfrak{f})$. A direct consequence of the Bratteli-Kishimoto-Robinson theorem (see Proposition 5.4.20 of [30]) is the following:

Proposition 1.14 ([P8]). *Assume that $\iota \equiv 0$ on $\mathcal{A}_e(\mathfrak{f})$, that $\mathfrak{f}$ is dense in $\mathfrak{h}$ and L^1-asymptotically Abelian for τ_0, and that ρ_0 is a factor. Then ρ is a KMS state.*

We will not define here what a KMS state is: it is enough to say that, when the fermionic system is constituted of ℓ baths, initially at thermal equilibrium at inverse temperatures $\beta_1, \dots, \beta_\ell$, then the global state ρ is KMS if and only if $\beta_1 = \dots = \beta_\ell$. In other words, under the assumptions of Proposition 1.14, the fluctuation algebra is commutative if and only if the original system is at thermal equilibrium. But then the fluctuation-dissipation theorem is precisely concerned with fluctuations at equilibrium, which is the case $\beta_1 = \ldots = \beta_\ell$. Theorem 1.11 shows that for all i, j, and for all Borel sets E, F (assuming if $L(\Phi_i, \Phi_j) = 0$ that 0 is not on the boundary of E):

$$\lim_{t\to\infty} \rho\big(\mathbb{1}_E(\tilde{\Phi}_{i,t})\mathbb{1}_F(\tilde{\Phi}_{j,t})\mathbb{1}_E(\tilde{\Phi}_{i,t})\big) = \mathbb{P}(N_i \in E, N_j \in F)$$

where $(N_1, \ldots, N_\ell)$ is a centered Gaussian vector with covariance matrix $\big(2L(\Phi_i, \Phi_j)\big)_{i,j=1}^{\ell}$. Any sequential measurement of heat fluctuations of two reservoirs is therefore given asymptotically by the distribution of of a pair of Gaussian random variables with covariance $2L$, where L is the matrix of Kubo transport coefficients given by (2). This seems to complete the proof of linear response for fermionic systems with local interactions.

Remark 1.15. One can then wonder why we did not continue with this program, for e.g. bosonic systems. A first reason is technical: the resummation techniques used in the "diagrammatic" proof are very specific to the fermionic case. A second and much more important reason is that our choice of fluctuations operators $\tilde{\Phi}_{i,t}$ is dictated by the relation

$$\tau^t(H_i) - H_i = \int_0^t \tau^s(\Phi_i)\,\mathrm{d}s$$

where the H_i are the Hamiltonians of the free dynamics of $\mathcal{R}_i$; in other words, we have investigated the fluctuations of variations $\tau^t(H_i) - H_i$. As we are about to show in the next chapter, these operators have no satisfactory physical meaning and have the bad taste of not satisfying some physically crucial symmetry properties. The discovery of another, physically sound, interpretation of fluctuations, led us to take another route. This new interpretation is presented in the next chapter.

Chapter 2

Two-time measurements and statistical formulations of thermodynamics

In the present chapter, we discuss the results of [P11] and [P21], and introduce the framework for part of the questions studied in [P22]. Note that even though [P11] was published as conference proceedings, it was the first time the connection between two-time measurements and relative modular operators was established, therefore introducing an efficient method to consider the limits (both thermodynamic and large-time limits) of two-time measurements statistics. However, because the technical aspects consist in general in applications of well-known procedures, we did not see it fit to make a journal publication out of these notes.

1. Insufficiency of the naive quantification

Until now, if A was an observable representing a certain physical quantity, we have modeled the variations of this quantity by the observable $\tau^t(A) - A$. In particular, when considering an out-of-equilibrium open quantum system, we have modeled the variation of the heat of reservoir $\mathcal{R}_j$ between times 0 and t by the observable $\tau^t(H_j) - H_j$. There are however two problems with this approach. The first lies in the operational meaning of this observable: we do not know of an experimental procedure allowing to measure it. The second problem is that, when one applies this approach to an entropy observable (see below), the derived distributions do not satisfy the *fluctuation relation*, which is a fundamental identity satisfied by classical systems. In the classical case, this relation was first observed in numerical simulations in [55], a first theoretical explanation was given in [57] and [64], and the first experimental observations were described in [123] (we refer the reader to [114] for a historical account of fluctuation relations).

To describe this relation, let us first consider the classical case; to avoid a change of notation, we consider a commutative C*-algebra $\mathcal{O}$, $(\tau^t)_t$ an automorphism group which is a Hamiltonian flow $\tau^t : f \mapsto f \circ \mathrm{e}^{t\mathcal{L}}$, and ρ a state which is now the integral with respect to a probability measure, which we suppose admits a density with respect to a $\mathcal{L}$-invariant reference measure (in practice, this reference measure is the Liouville measure of phase space). We then define for $t \in \mathbb{R}$ a random variable Σ_t by $\frac{\mathrm{d}\rho_{-t}}{\mathrm{d}\rho} = \mathrm{e}^{-t\Sigma_t}$. The variable Σ_t therefore represents the relative information per unit of time of ρ with respect to ρ_{-t} (see [35]), and satisfies relation $S(\rho_t|\rho) = t\rho(\Sigma_t)$. This identity (as well the similar relation for Rényi relative entropy, see (3) below) justifies seeing Σ_t as the entropy production random variable. We assume in addition that the system is time-reversal invariant, that is, there exists an involution ϑ on state space with $\{f \circ \vartheta, g \circ \vartheta\} = -\{f, g\}$, such that $\mathrm{e}^{t\mathcal{L}} \circ \vartheta = \vartheta \circ \mathrm{e}^{-t\mathcal{L}}$ and $\rho \circ \vartheta = \rho$. We will study the distributions $\mathbb{P}_t$ of $+\Sigma_t$ and $\widehat{\mathbb{P}}_t$ of $-\Sigma_t$, in both cases with respect to a reference state ρ. We show easily that $\Sigma_t \circ \mathrm{e}^{-t\mathcal{L}} = \Sigma_{-t} = -\Sigma_t \circ \vartheta$. For any bounded continuous function f, we therefore have by elementary manipulations

$$\rho\big(f(-\Sigma_t)\big) = \rho_t\big(f(-\Sigma_t) \circ \mathrm{e}^{-t\mathcal{L}}\big) = \rho_t\big(f(\Sigma_t) \circ \vartheta\big) = \rho_{-t}\big(f(\Sigma_t)\big) = \rho\big(f(\Sigma_t)\,\mathrm{e}^{-t\Sigma_t}\big). \quad (1)$$

As a consequence, $\mathbb{P}_t$ and $\widehat{\mathbb{P}}_t$ verify the universal relation

$$\frac{\mathrm{d}\widehat{\mathbb{P}}_t}{\mathrm{d}\mathbb{P}_t}(\varsigma) = \mathrm{e}^{-t\varsigma} \quad (2)$$

which shows that a decrease of entropy by a value ς is penalised (in the probabilistic sense) by a factor $\mathrm{e}^{-t\Sigma}$ with respect to an increase by the same amount. This says in particular that the probability of trajectories leading to a decrease of entropy is nonzero, even though this value is very small (this, including (2), was verified experimentally, see [123]).

Last, remark that it suffices to check identity (1) above in the case $f(s) = \mathrm{e}^{\alpha ts}$, so that (2) is equivalent to the invariance of $\alpha \mapsto \int \mathrm{e}^{-\alpha x}\,\mathrm{d}\mathbb{P}_t(x)$ by $\alpha \to 1-\alpha$. It is immediate that in the case where one has time reversal invariance, the Rényi relative entropy reads

$$S_\alpha(\rho_t|\rho) = \log \rho(\mathrm{e}^{-t\alpha\Sigma_t}). \quad (3)$$

As a consequence, relation (2) is equivalent to

$$S_\alpha(\rho_t|\rho) = S_{1-\alpha}(\rho_t|\rho). \quad (4)$$

Let us now return to the quantum case: assume $\mathcal{H}$ is a Hilbert space of finite dimension , and $\mathcal{O}$ is $\mathcal{B}(\mathcal{H})$ equipped with a faitfhul state ρ and a dynamics $\tau^t(X) = \mathrm{e}^{+\mathrm{i}tH} X \mathrm{e}^{-\mathrm{i}tH}$. If we define the observable Σ_t by $\tau^t(\log\rho) - \log\rho = -t\Sigma_t$, we again have[1]

$$t\rho(\Sigma_t) = S(\rho_t|\rho) \quad (5)$$

(where the relative entropy of two states σ and τ is essentially defined[2] by $S(\sigma|\tau) = \mathrm{tr}\,\big(\sigma(\log\sigma - \log\tau)\big)$) and a similar identity for Rényi relative entropy, see below. A relative entropy being positive, one has $\rho(\Sigma_t) \geq 0$, which is the usual formulation of the second principle of thermodynamics. In addition, in the case where ρ is a product of Gibbs states, i.e. is proportional to $\exp - \sum_j \beta_j H_j$, with the H_j mutually commuting observables (this will be the case when ρ describes the decoupled state of a number of reservoirs, each of them at thermal equilibrium), then $\Sigma_t = -\frac{1}{t}\sum_j \beta_j\big(\tau^t(H_j) - H_j\big)$; however $\sum_j \beta_j\big(\tau^t(H_j) - H_j\big)$ should then represent the total variation of (Clausius) entropy of the reservoirs, if we continue to view the variation of the energy of $\mathcal{R}_j$ as represented by $\tau^t(H_j) - H_j$. It is therefore expected, by analogy with the classical case, that the distributions of $+\Sigma_t$ and $-\Sigma_t$ in the state ρ satisfy relations identical to those we have observed in the classical case.

Let us therefore see if this is the case. One shows easily that $\tau^{-t}(\Sigma_t) = \Sigma_{-t} = -\vartheta(\Sigma_t)$. Once again, for any bounded continuous function f,

$$\rho\big(f(-\Sigma_t)\big) = \rho_t\big(\tau^{-t}\big(f(-\Sigma_t)\big)\big) = \rho_t\big(\vartheta\big(f(\Sigma_t)\big)\big) = \rho_{-t}\big(f(\Sigma_t)\big);$$

---

[1] One should be careful here more than anywhere else that if the state ρ has density matrix ϱ, then $\rho_t := \rho \circ \tau^t$ has density matrix $\tau^{-t}(\varrho)$.

[2] If one assumes for example that σ and are τ faitful; this is not a serious restriction.

again one can write

$$\rho_{-t}\big(f(\Sigma_t)\big) = \rho\big(f(\Sigma_t)\,\mathrm{e}^{\log\tau^t(\rho)}\mathrm{e}^{-\log\rho}\big) = \rho\big(f(\Sigma_t)\,\mathrm{e}^{\log\rho - t\Sigma_t}\mathrm{e}^{-\log\rho}\big)$$

but this is not in general equal to $\rho\big(f(\Sigma_t)\,\mathrm{e}^{-t\Sigma_t}\big)$. More precisely, if that relation was true for $f \equiv 1$, then one would have $\rho(\mathrm{e}^{-t\Sigma_t}) = 1$, and then

$$1 = \mathrm{tr}(\mathrm{e}^{\log\rho}\mathrm{e}^{-t\Sigma_t}) \geq \mathrm{tr}(\mathrm{e}^{\log\rho - t\Sigma_t}) = \rho_t(\mathrm{Id}) = 1$$

where the upper bound is a direct application of the Golden–Thompson inequality (see example IX.3.7 of [21]). Yet the Golden–Thompson inequality is saturated only if the operators involved, here $\log\rho$ and $-t\Sigma_t$, commute (see Proposition II.7.13 of [105]), which is equivalent to the commutation of ρ and H. As a consequence, relation $\frac{\mathrm{d}\widehat{\mathbb{P}}_t}{\mathrm{d}\mathbb{P}_t}(\varsigma) = \mathrm{e}^{-t\varsigma}$ does not hold unless $\rho_t \equiv \rho$ for all t, which strongly limits the interest of the considered model. In addition, any interpretation in terms of trajectory is a priori impossible, since the notion of trajectory is problematic in the quantum case (see [113]).

A new approach is therefore required if one hopes to recover universal symmetries like (1). Remark that, in the classical case (1) is equivalent to the symmetry (4) of Rényi relative entropy. The non-commutative Rényi relative entropy is defined essentially[3] by $S_\alpha(\rho_t|\rho) = \log\mathrm{tr}(\rho_t^\alpha\rho^{1-\alpha})$; we will say that the system defined by the space $\mathcal{H}$, the state ρ and the Hamiltonian H is *time-reversal invariant* (TRI) if there exists an antilinear involution ϑ of $\mathcal{B}(\mathcal{H})$ such that $\rho\big(\vartheta(A)\big) = \rho(A^*)$ for all $A \in \mathcal{B}(\mathcal{H})$, and $\vartheta(H) = H$.

Under this TRI assumption, one has using the antilinearity of ϑ, then the invariance of the trace par τ^t:

$$\mathrm{tr}(\rho_t^\alpha\rho^{1-\alpha}) = \mathrm{tr}\,\big(\vartheta(\rho^{1-\alpha}\rho_t^\alpha)\big) = \mathrm{tr}(\rho^{1-\alpha}\rho_{-t}^\alpha) = \mathrm{tr}(\rho_t^{1-\alpha}\rho^\alpha).$$

The functional $S_\alpha(\rho_t|\rho)$ therefore satisfies the symmetry with respect to $\alpha \mapsto 1-\alpha$ if the system is TRI. One does not, however, know how to interpret this symmetry in an "operational" way. This is what we discuss in the next section.

2. Relative modular operator and two-time measurements

At the beginning of the years 2000, two prepublications suggested new functionals that verified relations of the type $e(\alpha) = e(1-\alpha)$, and that were related to projective measurements. The first we will mention is due to Matsui and Tasaki [120]. To discuss it, we start by defining, in the case where $\mathcal{O} = \mathcal{B}(\mathcal{H})$ with $\dim\mathcal{H} < \infty$, its *standard representation*: on the Hilbert space $\mathcal{K} := \mathcal{B}(\mathcal{H})$ equipped with the Hilbert–Schmidt scalar product $\langle A, B\rangle_{\mathcal{K}} := \mathrm{tr}(A^*B)$, we define for ν a state of $\mathcal{O}$ the vector $\Omega_\nu = \nu^{1/2}$ of $\mathcal{K}$, and if μ and ν are two faithful states, a self-adjoint operator $\Delta_{\mu|\nu}$ on $\mathcal{K}$ by

$$\Delta_{\mu|\nu}(X) = \mu X \nu^{-1}.$$

Matsui and Tasaki suggest considering the Laplace transform of the distribution of the operator $\log\Delta_{\rho_t|\rho}$ in the pure state $|\Omega_{\rho_t}\rangle\langle\Omega_{\rho_t}|$. This distribution $\mathbb{P}_t$ then satisfies the invariance of $\alpha \mapsto \int \mathrm{e}^{-\alpha x}\,\mathrm{d}\mathbb{P}_t(x)$ under $\alpha \mapsto 1-\alpha$: this amounts to an identity observed previously, since by a direct calculation

$$\langle\Omega_{\rho_t}, \mathrm{e}^{-\alpha\log\Delta_{\rho_t|\rho}}\,\Omega_{\rho_t}\rangle = \langle\Omega_{\rho_t}, \Delta_{\rho_t|\rho}^{-\alpha}\,\Omega_{\rho_t}\rangle = \exp S_{1-\alpha}(\rho_t|\rho).$$

[3] Again, this expression is rigorous if one supposes that both states faithful.

The proposal of Tasaki and Matsui therefore did relate the desired symmetry to the distribution of a self-adjoint operator, but the latter was not an observable in the sense that it did not proceed from the initial algebra of observables: the operator $\Delta_{\rho_t|\rho}$ does not belong to $\pi(\mathcal{O})$ (nor to the generated von Neumann algebra $\pi(\mathcal{O})''$). The physical meaning of this symmetry is therefore not clear; this proposal, however, has the advantage of extending immediately to a general von Neumann algebra (the article [120] considered in this general framework).

The second proposal we will discuss is that of Kurchan, in the prepublication [89] (which was never published). It is based on what we now call *two-time measurements statistics*[4] Because this notion will appear repeatedly, let us consider a more general framework than that of Kurchan (i.e. for other quantities than just entropy): we therefore return to $\mathcal{O} = \mathcal{B}(\mathcal{H})$ equipped with the unitary evolution defined by the Hamiltonian H, and consider an observable A. One can then perform the following experiment:

- measure A at time 0, which gives a random outcome a_1 and modifies the state of the system following the projection postulate (1);
- evolve the system for a time t;
- measure again A at time t, which gives a random outcome a_2.

The two-time measurements statistics of A between times 0 and t is defined as the distribution of $\varsigma = (a_2 - a_1)/t$. It therefore has a perfectly defined operational meaning, but relies in a crucial way on the finite dimension of the system.

Remark 2.1. It would make sense to measure a different observable before and after the evolution, but we restrict ourselves to the situations where we measure the same observable twice.

One can write explicitly the distribution of ς: for this, write the spectral decomposition of A as $A = \sum_{a\in \mathrm{sp}\, A} a\, \pi_a$. According to the Born rule, a measurement of A in an initial state ρ gives an outcome a_1 with probability $\mathrm{tr}(\rho\, \pi_{a_1})$, after which the the system is in the state $\pi_{a_1}\rho\pi_{a_1} / \mathrm{tr}(\rho\pi_{a_1})$. One can also write these quantities under the form $\mathrm{tr}(\tilde\rho_A\pi_{a_1})$ and $\tilde\rho_A\pi_{a_1} / \mathrm{tr}(\tilde\rho_A\pi_{a_1})$, where $\tilde\rho_A$ is *the a priori state* relative to A, defined as

$$\tilde\rho_A = \sum_{a\in \mathrm{sp}\, A} \pi_a\rho\pi_a. \tag{6}$$

In most situations (in particular that considered by Kurchan, see below), ρ and A will commute with one another so that $\rho = \tilde\rho_A$. After this first measurement, the system evolves for a time t and its state becomes $\mathrm{e}^{-\mathrm{i}tH}\tilde\rho_A\pi_{a_1}\mathrm{e}^{+\mathrm{i}tH} / \mathrm{tr}(\tilde\rho_A\pi_{a_1})$. The second measurement of A, at time t, gives the outcome a_2 with probability conditional on a_1

$$\mathrm{tr}\left(\mathrm{e}^{-\mathrm{i}tH}\tilde\rho_A\pi_{a_1}\mathrm{e}^{+\mathrm{i}tH}\pi_{a_2}\right) / \mathrm{tr}(\tilde\rho_A\pi_{a_1}).$$

The probability to obtain a_1 then a_2 in these two measurements is therefore

$$\mathrm{tr}\left(\mathrm{e}^{-\mathrm{i}tH}\tilde\rho_A\,\pi_{a_1}\mathrm{e}^{+\mathrm{i}tH}\pi_{a_2}\right).$$

The two-time measurements statistics of A is the distribution of $\phi = (a_2 - a_1)/t$, so that

$$\mathbb{P}_{A,t}(\phi) = \sum_{a_2-a_1=t\phi} \mathrm{tr}\left(\mathrm{e}^{-\mathrm{i}tH}\tilde\rho_A\,\pi_{a_1}\mathrm{e}^{+\mathrm{i}tH}\pi_{a_2}\right).$$

[4] The latter is still called for historical reasons the "full counting statistics" because its first appearance, due to Lesovik and Levitov (in [92]) considered numbers of electrons going through a circuit.

This expression of $\mathbb{P}_{A,t}$ seems impractical, but a magical simplification applies when one considers the generating function $\chi_{A,t}(\alpha) = \int \mathrm{e}^{-\alpha t\phi}\, \mathrm{d}\mathbb{P}_{A,t}(\phi)$: one sees immediately that

$$\chi_{A,t}(\alpha) = \mathrm{tr}\,\big(\mathrm{e}^{-\mathrm{i}tH}\tilde{\rho}_A\, \mathrm{e}^{+\alpha A}\mathrm{e}^{+\mathrm{i}tH}\mathrm{e}^{-\alpha A}\big). \tag{7}$$

This construction extends immediately to the case where one measures more than one observable $A_1,\ldots,A_\ell$: if they commute, they can be simultaneously measured and one obtains a distribution $\mathbb{P}_{\mathbf{A},t}$ on $\mathbb{R}^\ell$, satisfying (7) with $\alpha \in \mathbb{R}^\ell$ and $\alpha \cdot \mathbf{A} := \sum_{j=1}^{\ell} \alpha_k A_k$ in place of αA.

Remark 2.2. An immediate consequence of (7) is:

$$\begin{aligned} -\frac{\partial\chi_{A,t}}{\partial\alpha}\Big|_{\alpha=0} &= \int \phi\, \mathrm{d}\mathbb{P}_{A,t}(\phi) &= \tilde{\rho}_A\big(\tau^t(A) - A\big), \\ \frac{\partial^2\chi_{A,t}}{\partial\alpha^2}\Big|_{\alpha=0} &= \int \phi^2\, \mathrm{d}\mathbb{P}_{A,t}(\phi) &= \tilde{\rho}_A\big((\tau_t(A) - A)^2\big), \end{aligned} \tag{8}$$

which shows that the first two moments of $\mathbb{P}_t$ are the same as those of the distribution of $\tau^t(A) - A$ in the state ρ. This helps to understand why so many studies of out-of-equilibrium situations were conducted at the level of these two moments, and why the two-time measurements formalism took so long to emerge.

Let us now come back to the proposal of Kurchan in [89], where one considers the case $A = -\log\rho$. One then has $\tilde{\rho}_A = \rho$, and the functional $e_t := \log\chi_{A,t}$ can be expressed as

$$e_t(\alpha) = \log \mathrm{tr}\,\big(\mathrm{e}^{-\mathrm{i}tH}\rho^{1-\alpha}\mathrm{e}^{+\mathrm{i}tH}\rho^{+\alpha}\big) = \log \mathrm{tr}\,\big(\rho^{\alpha/2}\mathrm{e}^{-\mathrm{i}tH}\rho^{1-\alpha}\mathrm{e}^{+\mathrm{i}tH}\rho^{\alpha/2}\big). \tag{9}$$

One remarks that this last expression gives

$$e_t(\alpha) = \log \mathrm{tr}\,\big(\rho_t^{1-\alpha}\rho^{\alpha}\big) = S_{1-\alpha}(\rho_t|\rho).$$

As a consequence, the functionals suggested by Tasaki and Matsui in [120] on the one hand, by Kurchan in [89] on the other, are essentially the same. Two-time measurements therefore give access to the distribution of quantities which are not mathematical observables. One can equivalently define a probability $\mathbb{P}_t$ as the two-time measurements statistics of $A = -\log\rho$ or as the distribution of $\frac{1}{t}\log\Delta_{\rho_t|\rho}$ in the pure state $|\Omega_{\rho_t}\rangle\langle\Omega_{\rho_t}|$, and then $\mathbb{P}_t$ satisfies property (2). This observation was done for the first time in [P11], and allows to extend to a more general framework the construction of two-time measurements (see the next section).

Remark 2.3. It is worthwhile at this point to discuss the difference between the two approaches that use respectively two-time measurements, and sequential measurements as was done in the preceding chapter. An obvious difference is that the distribution $\mathbb{P}_{A,t}$ of two-time measurements of A is the law of a variation $a_2 - a_1$. One could think that it is simply a marginal of the sequential measurements statistics, of A first then of its evolution $\mathrm{e}^{+\mathrm{i}tH}A\mathrm{e}^{-\mathrm{i}tH}$. Recall that the probability to obtain two measurements a_1, then a_2 is

$$\mathrm{tr}\,\big(\mathrm{e}^{-\mathrm{i}tH}\pi_{a_1}\rho\pi_{a_1}\mathrm{e}^{+\mathrm{i}tH}\pi_{a_2}\big)$$

which corresponds to the expression of the probability of a sequential measurement (a_1, a_2) of A then $\mathrm{e}^{+\mathrm{i}tH}A\mathrm{e}^{-\mathrm{i}tH}$ (see (1.6)). However, summing this last expression over all $a_1 \in E$ does not in general give the probability of a sequential measurement, in E and then equal to b (it would be the case if ρ commuted with A: remark that in this respect, the introduction of $\tilde{\rho}_A$ above tends to obfuscate this point). The protocol defining two-time measurements does not assume in general that one has access to all possible "questions" concerning A during the first measurement, but that one makes a measurement of the value of the observable A. This is what makes the operational definition of two-time measurements beyond finite dimension difficult.

Remark 2.4. The article [P11] actually defines in the framework of $\mathcal{O} = \mathcal{B}(\mathcal{H})$, $\dim \mathcal{H} < \infty$, a one-parameter family of functionals: for all $p \geq 1$, one defines

$$e_{p,t}(\alpha) = \log \operatorname{tr} \left(\left(\rho^{(1-\alpha)/p} \rho_t^{2\alpha/p} \rho^{(1-\alpha)/p} \right)^{p/2} \right),$$

so that the above functional e_t is $e_t = e_{2,t}$. Since $\log \rho_t = \log \rho + t\Sigma_{-t}$, the Lie–Trotter formula shows that

$$\lim_{p\to\infty} e_{p,t}(\alpha) = \log \operatorname{tr}(\mathrm{e}^{\log \rho + \alpha t \Sigma_{-t}}) =: e_{\infty,t}(\alpha).$$

The Araki–Lieb–Thirring inequality (inequality (IX.13) in [21]) shows that $p \mapsto e_{p,t}(\alpha)$ is a continuous nonincreasing function. One shows easily that $\alpha \mapsto e_{p,t}(\alpha)$ is convex, real-analytic, and verifies under the time-reversal assumption the symmetry

$$e_{p,t}(1-\alpha) = e_{p,t}(\alpha) \text{ for } \alpha \in \mathbb{R}.$$

These functionals $e_{p,t}(\alpha)$ have been rediscovered and studied under the name of (α, z)-entropies by Audenaert and Datta ([10]) and the functions $\alpha \mapsto e_{2\alpha,t}(\alpha)$ (that do not satisfy the symmetry under $\alpha \mapsto 1-\alpha$) are the "sandwiched Rényi entropies" defined later on in [103,124], that have applications for strong converses of quantum hypothesis testing, see [75,101,102]. Since the functionals $e_{p,t}$ for $p \neq 2$ do not have, as far as we know, any operational interpretation (except the quite indirect interpretation given by those strong converses), we will not mention them any further in this document.

3. Statistical formulation of the second principle

The previous section gave, in the setup of $\mathcal{O} = \mathcal{B}(\mathcal{H})$ with $\dim \mathcal{H} < \infty$ a probability distribution $\mathbb{P}_t$ for the rate of variation of entropy between 0 and t. However, we are mostly interested in the regime $t \to \infty$, and many arguments show that essentially nothing happens if $\mathcal{H}$ remains finite-dimensional (one can for example prove that $S(\rho_t|\rho) \underset{t\to 0}{\longrightarrow} 0$ if H has discrete spectrum, see remark 5.1 of [P11]), which shows that any NESS ρ_+ is normal – normality being the property analogous to absolute continuity – with respect to ρ). As a consequence, before considering the limit $t \to \infty$, we must take the *thermodynamic limit* of the system, to make it "infinite".

A first approach is to consider from the start an operator algebra describing this infinite system. It is of course possible, and as we write, one of the interests of the connection with the definition by Matsui and Tasaki of the two-time measurements statistics is that it extends to that case, see remark 2.9. One can however avoid introducing the algebraic formalism by modeling the infinite system as approximated (in a sense that will depend on the models) by a sequence $(\mathcal{H}^{(N)}, \rho^{(N)})_N$ of finite spaces equipped with a state, and assuming that the sequence of functions $e_t^{(N)}$ associated with $(\mathcal{H}^{(N)}, \rho^{(N)})$ by (9) converges in a sufficient way. We will assume that for all $t \in \mathbb{R}$,

$$e_t(\alpha) := \lim_{N\to\infty} e_t^{(N)}(\alpha) \text{ exists for } |\alpha| < \alpha_0. \tag{10}$$

Under that assumption, there exists for all t a probability distribution $\mathbb{P}_t$ on $\mathbb{R}$ which is the limit in the sense of exponential moments of the sequence $(\mathbb{P}_t^{(N)})_N$ where $\mathbb{P}_t^{(N)}$ is associated with $(\mathcal{H}^{(N)}, \rho^{(N)})$. We have in particular by (8) and (10)

$$\int \varsigma \, \mathrm{d}\mathbb{P}_t(\varsigma) = \lim_{N\to\infty} \rho^{(N)}(\Sigma_t^{(N)}) = \lim_{N\to\infty} S(\rho_t^{(N)} | \, S^{(N)}) \tag{11}$$

which is therefore a positive quantity. We will see $\mathbb{P}_t$ as the law of the rate of variation of entropy between times 0 and t: relation (11), which is an expression of the second principle of thermodynamics, tells us that the *average* variation of entropy is nonnegative. If we assume in addition that the system is time-reversal invariant (in the sense that all systems $\mathcal{H}^{(N)}, \rho^{(N)}, H^{(N)}$ are), the symmetry $e_t^{(N)}(\alpha) = e_t^{(N)}(1-\alpha)$ implies that the convergence (10) holds on an open real neighbourhood of $[0, 1]$, and one still has

$$e_t(\alpha) = e_t(1-\alpha) \tag{12}$$

on that open neighbourhood. This symmetry is called "Evans–Searles symmetry" (in reference to [57]), and implies the relation

$$\frac{\mathrm{d}\widehat{\mathbb{P}}_t}{\mathrm{d}\mathbb{P}_t}(s) = \mathrm{e}^{-ts} \tag{13}$$

which expresses the relative probabilities of an increase of entropy, or a decrease by the same amount, on $[0, t]$. This result on the distribution $\mathbb{P}_t$ is stronger than the positivity of the average: it implies that $\int_{\mathbb{R}} \mathrm{e}^{-ts} \, \mathrm{d}\mathbb{P}_t = \int_{\mathbb{R}} \mathrm{d}\mathbb{P}_t = 1$ (a relation analogous to the Jarzynski identity) which by the Jensen inequality implies in turn $\int_{\mathbb{R}} s \, \mathrm{d}\mathbb{P}_t \geq 0$. We view relation (13) as a statistical expression of the second principle of thermodynamics, which applies to individual trajectories instead of averages over numerous realizations. Note also that an advantage of fluctuation relations is that they give equalities (such as $\int_{\mathbb{R}} \mathrm{e}^{-ts} \, \mathrm{d}\mathbb{P}_t = 1$) whereas formulations for averages typically give inequalities (as $\int_{\mathbb{R}} s \, \mathrm{d}\mathbb{P}_t \geq 0$).

Remark 2.5. One can object that in the limit $N \to \infty$, our experimental protocol would require to measure a non local quantity (entropy or, further below, energy of an infinite reservoir), and this does not seem feasible in practice. Experimental processes based on the observation of a single auxiliary spin have been proposed by [33, 50, 98] and implemented by [15].

It remains, in order to extract thermodynamical information from our functionals, to consider the limit $t \to \infty$. Let us define without anticipating on their domain of definition,

$$e_+(\alpha) := \lim_{t\to\infty} \frac{1}{t} \, e_t(\alpha). \tag{14}$$

We can then give the following result:

Proposition 2.6 ([P11]). *Assume that (14) holds for all α in an open interval $\mathcal{I}$ and that the limiting function e_+ is differentiable on $\mathcal{I}$. Then:*

1. *if $\mathcal{I}$ is a neighbourhood of the origin, then ς converges exponentially to $\langle\sigma\rangle_+ := -e'_+(0) \geq 0$;*
2. *if there exists a complex neighbourhood $\mathcal{V}$ of the origin such that $\sup_{t>1} \sup_{\alpha\in\mathcal{V}} \frac{1}{t} |\log e_t(\alpha)| < \infty$, then one has convergence in distribution*

$$\sqrt{t}\big(\varsigma - \langle\sigma\rangle_+\big) \underset{t\to\infty}{\longrightarrow} \mathcal{N}\big(0, e''_+(0)\big) \, ;$$

3. *if one assumes that $\mathcal{I}$ contains $[0,1]$ (we then also assume that $\mathcal{I}$ is symmetric with respect to $1/2$ to simplify the notation), and that the system is time-reversal invariant, then*

$$e_+(\alpha) = e_+(1-\alpha) \text{ for all } \alpha \in \mathcal{I}, \tag{15}$$

and with $S := \sup_{\alpha\in\mathcal{I}} e'_+(\alpha)$, we have that

$$I(\varsigma) := -\inf_{\alpha\in\mathcal{I}} \big(\alpha\varsigma + e_+(\alpha)\big)$$

is nonnegative, convex, differentiable on $(-S,+S)$, vanishes only at $\langle\sigma\rangle_+$, and satisfies

$$I(-s) = s + I(s). \tag{16}$$

Then, ς satisfies a local large deviations principle: for example, for any open subset J of the interval $(-S,+S)$:

$$\lim_{t\to\infty} \log \mathbb{P}_t(J) = -\inf_{s\in J} I(s). \tag{17}$$

This result follows immediately from the Gärtner-Ellis Theorem (see Theorems 2.3.6 of [45] and Theorems II.6.3–II.6.4 of [52]), Bryc's Theorem (see [32]) and finite-time symmetries (12).

Remark 2.7. Relation (16), satisfied by the rate function I, is inherited from symmetry (12). Remark however that the large deviations principle (17) does not tell us anything new, since it is a consequence of the finite-time relation (13), which is exact. The other two points are consequences of the assumption that e_+ exists, with no direct relation to the symmetries. We have only stated Proposition 2.6 to refer to it later on.

Remark 2.8. Symmetry $e_+(\alpha) = e_+(1-\alpha)$ is a priori not the same as the *Gallavotti–Cohen symmetry* (see [64]). The latter is also expressed as a symmetry relation $e_{\mathrm{GC}}(\alpha) = e_{\mathrm{GC}}(1-\alpha)$ for a certain functional e_{GC}. The differences between e_+ and e_{GC} lies in the reference state they consider: the initial state ρ in the case of e_+, and the steady state ρ_+ for e_{GC}. In all non-trivial models for which we have been able to compute both functionals, however, they turned out to be equal.

Remark 2.9. The downside of our approach of the thermodynamic limit via the existence of limits to the functionals $e_t^{(N)}$ is that it prevents us from giving any expression in terms of the "true" dynamical system. We will therefore discuss the other approach, that uses the definition by Matsui and Tasaki of the functional e_t. The latter uses the Tomita–Takesaki theory, which shows that any von Neumann algebra $\mathcal{O}$ admits a *standard representation* $\pi : \mathcal{O} \to \mathcal{B}(\mathcal{K})$ (see section 2.5.4 of [29]), unique up to unitary conjugation, such that there exist a self-adjoint operators $\Delta_{\rho_t|\rho}$ on $\mathcal{K}$ which have a number of remarkable properties, and vectors Ω_η for any state η, such that $\langle\Omega_\eta, \pi(X)\Omega_\eta\rangle_{\mathcal{K}} = \eta(X)$. One can then define the functional

$$e_t(\alpha) := \log\langle\Omega_\rho, \Delta^\alpha_{\rho_t|\rho}\Omega_\rho\rangle_{\mathcal{K}}$$

and the probability $\mathbb{P}_t$ as the distribution of the operator $-\frac{1}{t}\log\Delta_{\rho_t|\rho}$ in the pure state $|\Omega_\rho\rangle\langle\Omega_\rho|$. In addition, in many situations where the physical system under consideration $\mathcal{O}$ can be approximated by finite systems $\mathcal{O}^{(N)}$, one can show that the associated functionals $e_t^{(N)}(\alpha)$ converge for all t and α of $\mathbb{R}$, to that e_t. For spin systems or fermionic systems, this can be done by standard techniques developed in the seventies (see for example [30], as well as sections 6.5 to 6.7 of [P11]). In that case, one can show for example that

$$\langle\sigma\rangle_+ = \lim_{t\to\infty}\frac{1}{t}S(\rho_t|\rho) = \lim_{t\to\infty}\frac{1}{t}\int_0^t \rho(\sigma_s)\,\mathrm{d}s$$

where σ is a certain observable called *entropy flux observable* that, when the system is multi-thermal, satisfies $\sigma = \sum_i \beta_i \Phi_i$ where Φ_i is the observable for the flux coming out of $\mathcal{R}_i$, mentioned at the beginning of chapter 1.

Let us now discuss the methods of proof of convergence (14), and of the properties of e_+. One can in certain cases like fermionic quasi-free systems or the XY spin chain (that can be mapped to the quasi-free case by a Jordan–Wigner transform), obtain explicit formulas for $e_t^{(N)}(\alpha)$ using trace formulas, and as a consequence follow our program of discussing the thermodynamic limit leading to e_t, and the large-time limit leading to e_+, only at the level of these functionals. The final expression of e_+ can then be written in terms of the one-particle wave operator: see sections 6.6 and 6.7 of [P11]. The relevant methodes to take care of these limiting procedures are generally the spectral tools developed for extended systems; it is therefore beneficial to start from the expression of e_t obtained in the algebraic framework of remark 2.9. One can then treat the spin-fermion case using the fact that e_t can be expressed in terms of the L^p-Liouvilleans of Araki-Masuda ([6]), these L^p-Liouvilleans L_p being defined as generators of groups $\mathrm{e}^{-\mathrm{i}tL_p}$ that implement the dynamics τ^t and are made of isometries of a non-commutative L^p space. One has indeed (it is once again immediate to prove in the case where $\mathcal{O} = \mathcal{B}(\mathcal{H})$ with $\dim \mathcal{H} < \infty$),

$$e_t(\alpha) = \log\langle \Omega_\rho, \mathrm{e}^{-itL_{\frac{1}{\alpha}}} \Omega_\rho\rangle_{\mathcal{K}}.$$

The advantage of this expression is that the $\mathrm{L}^{\frac{1}{\alpha}}$-Liouvillean of the free dynamics is simple, and that of the coupled dynamics, which is our $L_{\frac{1}{\alpha}}$, can be written as a perturbation of the former. One can then use complex deformations techniques to prove that $\lim_{t\to\infty} \frac{1}{t}\, e_t(\alpha)$ can be expressed from the resonances of $L_{\frac{1}{\alpha}}$. We refer to sections 5.5 and 6.5 of [P11], which are themselves based on [80].

Remark 2.10. We have therefore shown that a definition through two-time measurements of the variations of entropy leads to a statistical formulation of the second principle of thermodynamics: this variation of entropy can be negative, but with a probability which, compared to the probability of a positive variation by the same amount, is smaller by a factor exponential in this value (once again, this has been observed experimentally, see [123]). This comparison of probabilities of an increase or decrease of entropy is precisely expressed by the symmetry relation $e_+(\alpha) = e_+(1-\alpha)$ of generating functions.

4. Statistical formulation of the first principle

It is natural now to ask whether the first principle of thermodynamics can have a statistical formulation in the quantum framework thanks to two-time measurements statistics, and whether this formulation is summarized by a symmetry of generating functions.

The article [P12] answers the first question. To set the framework for the two-time measurement of the energy in a quantum system, let us again consider $\mathcal{O} = \mathcal{B}(\mathcal{H})$ with $\dim \mathcal{H} < \infty$, suppose that the system is equipped with a free dynamics described by a Hamiltonian H_0 and with an initial state ρ, invariant by the free dynamics, and that at time $t = 0$ an "exterior" force is turned on, so that the effective Hamiltonian becomes $H = H_0 + V$. We are interested in the law of the rate of variation of H_0 between times 0 and t, defined as the two-time measurement of H_0. We will denote by $t\phi_0$ this random variable and by $\mathbb{P}_t$ the associated probability (which should be denoted $\mathbb{P}_{H_0,t}$ in accordance with the notation of section 2). The first two moments of that distribution are, by Remark 2.2,

$$\rho\big(\tau^t(H_0) - H_0\big) \quad \text{et} \quad \rho\big(\big(\tau^t(H_0) - H_0\big)^2\big)$$

and because $H_0 + V$ is a conserved quantity, $\tau^t(H_0) - H_0 = \tau^t(V) - V$, and therefore

$$|\mathbb{E}_t(\phi_0)| \le \frac{2}{t}\|V\| \qquad \mathbb{E}_t\big((\phi_0 - \mathbb{E}_t(\phi_0))^2\big) \le \frac{2}{t^2}\|V\|^2 \tag{18}$$

so that if, in the thermodynamic limit, the coupling V remains uniformly bounded, then the first two moments of $t\phi_0$ are uniformly bounded in time. It is generally considered in the literature (see for example section III.B.3 of [54]) that under such an assumption on the coupling, all moments of ϕ_0 should tend to zero, or even that there should exist uniformly bounded exponential moments of $t\phi_0$. The article [P12] has identified conditions under which it is the case; the article [20] later showed that these conditions were actually necessary. Denote then $\xi_t(\theta) = \mathbb{E}_t(\mathrm{e}^{-\theta t\phi_0})$, which is given according to (7) by

$$\xi_t(\theta) = \mathrm{tr}(\mathrm{e}^{-\mathrm{i}tH}\rho\mathrm{e}^{+\theta H_0}\mathrm{e}^{+\mathrm{i}tH}\mathrm{e}^{-\theta H_0})\,;$$

one shows easily (see [P12]) by standard trace inequalities that if one defines

$$S_{H_0}(\theta_0) = \sup_{|\theta|\le|\theta_0|} \|\mathrm{e}^{+\frac{1}{2}\theta H_0}\, V\, \mathrm{e}^{-\frac{1}{2}\theta H_0}\|, \tag{19}$$

then

$$\mathrm{e}^{-2|\theta| S_{H_0}(\theta)} \le \xi_t(\theta) \le \mathrm{e}^{+2|\theta| S_{H_0}(\theta)}. \tag{20}$$

These bounds have an elementary consequence, that we formulate once again assuming that the final system is described through a sequence $(\mathcal{H}^{(N)}, \rho^{(N)}, H_0^{(N)})_N$ of finite spaces, each of them equipped with a state and a free Hamiltonian $H_0^{(N)}$. We denote with a subscript (N) all objects associated with the N-th finite system. The main result of [P12] is:

Theorem 2.11 ([P12]). *If there exists θ such that*

$$S_{H_0}(\theta) = \sup_N S_{H_0}^{(N)}(\theta) < \infty, \tag{21}$$

then

$$\sup_N \mathbb{E}_t(\mathrm{e}^{\theta t\phi_0^{(N)}}) \le 2\,\mathrm{e}^{2\theta S_{H_0}(\theta)}. \tag{22}$$

In particular, if one can define a limiting random variable ϕ_0 when $N \to \infty$ (for example if one has the convergence $\lim_{N\to\infty} \xi_t^{(N)}(\theta) =: \xi_t(\theta)$ for θ in a neighbourhood of zero), then this random variable ϕ_0 has exponential moments: $\mathbb{E}_t(\mathrm{e}^{\theta t\phi_0}) \le 2\,\mathrm{e}^{2\theta S_{H_0}(\theta)}$ for $|\theta|$ small enough. The above proof is mathematically elementary, but nevertheless identified an optimal condition: the article [20] shows examples in which (21) is necessary to have $\mathbb{E}_t(\mathrm{e}^{\theta t\phi_0}) < \infty$ (as well as examples in which $t\phi_0$ has no moments of order higher than 4). This suprising phenomenon can be understood from the fact that even a bounded V can induce transitions between energy levels of H_0 that are arbitrarily far from one another: in a Fermi golden rule approximation, the transitions between energy levels E and E' of H_0 with respective eigenvectors Ψ_E, $\Psi_{E'}$ induced by the interaction have rate $T(E,E') = |\langle\Psi_{E'}, V\Psi_E\rangle|^2$. The bound (19) therefore implies that transitions towards high energy have exponentially decreasing probabilities. For this reason, we view a condition such as $S_{H_0}(\theta_0) < \infty$ as an ultraviolet condition (and this is confirmed by the study of examples in [20]).

Let us now move on to the question of expressing the first principle through the symmetry of a generating function. One proposal in this direction was done by Andrieux, Gaspard, Monnai and Tasaki in [5].

This article however, has a mathematical flaw, and the article [P21] attempted to mend it; we however had to introduce an additional assumption, and the examples exhibited in [20] show that this assumption is necessary. In [P21], we suppose that the system is made of different reservoirs, and study the detailed energy variation, that is, the variation in each reservoir.

Once again, we begin with the finite-dimensional case. We therefore suppose, for $\mathcal{H}$ finite-dimensional, that we have a family $\boldsymbol{E} = (H_1, \ldots, H_\ell)$ of operators of $\mathcal{B}(\mathcal{H})$ that commute with one another, and V another observable. We will see each H_j as the "free Hamiltonian of the j-th reservoir $\mathcal{R}_j$", and V as a coupling operator, so that $H_0 = H_1 + \ldots + H_\ell$ is the free Hamiltonian of the full system, and $H = H_0 + V$ the Hamiltonian with interactions. We denote by ρ the initial state of the system, which we will in general suppose (see remark 2.17) is a multi-thermal state, i.e. is of the form:

$$\rho_{\boldsymbol{\beta}} = Z^{-1}\, \mathrm{e}^{-\sum_j \beta_j H_j} = Z^{-1}\, \mathrm{e}^{-\boldsymbol{\beta}.\boldsymbol{E}} \text{ with } Z = \mathrm{tr}(\mathrm{e}^{-\sum_j \beta_j H_j}) = \mathrm{tr}(\mathrm{e}^{-\boldsymbol{\beta}.\boldsymbol{E}}),$$

where $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_\ell)$ is an ℓ-tuple of inverse temperatures. Every time we suppose that the initial state ρ is multi-thermal, we will denote it by $\rho_{\boldsymbol{\beta}}$; in that case, the state ρ commutes with $\boldsymbol{E}$.

Since the H_j commute, it is possible to make a simultaneous measurement of the components of $\boldsymbol{E}$. The associated two-time measurements statistics then defines a $\mathbb{R}^\ell$-valued random variable $\boldsymbol{\phi}$, with distribution $\mathbb{P}_{\boldsymbol{E},t}$, and whose generating function $\chi_t(\boldsymbol{\alpha}) = \int e^{-\boldsymbol{\alpha}.\boldsymbol{\phi}}\, \mathrm{d}\mathbb{P}_{\boldsymbol{E},t}(\boldsymbol{\phi})$ is given according to (7) by

$$\chi_t(\boldsymbol{\alpha}) = \mathrm{tr}\left(\mathrm{e}^{-\mathrm{i}tH}\,\tilde{\rho}\,\mathrm{e}^{+\boldsymbol{\alpha}.\boldsymbol{E}}\mathrm{e}^{+\mathrm{i}tH}\mathrm{e}^{-\boldsymbol{\alpha}.\boldsymbol{E}}\right).$$

If one denotes $H_0 = H_1 + \ldots + H_\ell$, then with the above notation we have $\xi_t(\alpha) = \chi_t(\alpha\mathbf{1})$.

Remark 2.12. In the same way as with the previously considered variations of entropy in section 2, one can give definitions of χ_t and ξ_t in terms of modular objects. These definitions would be valid in an infinite system, and one can prove for different models that they are the thermodynamic limits of the functionals associated with the confined models.

We will be interested here in the relation between $\chi_t(\boldsymbol{\alpha})$ and $\chi_t(\boldsymbol{\alpha}+\theta\mathbf{1})$; we recall that $(\boldsymbol{\alpha}+\theta\mathbf{1}).\boldsymbol{E}$ equals $\boldsymbol{\alpha}.\boldsymbol{E} + \theta H_0$, which explains that this relation can express a form of conservation of energy. Once more we can obtain relevant information for confined systems from trace inequalities. One has (Proposition 2.6 of [P21]):

$$\mathrm{e}^{-2|t|S(\|\boldsymbol{\alpha}\|)} \leq \chi_t(\boldsymbol{\alpha}) \leq \mathrm{e}^{+2|t|S(\|\boldsymbol{\alpha}\|)} \tag{23}$$

where $\|\boldsymbol{\alpha}\| := \max|\alpha_j|$ and where for $\alpha_0 \in \mathbb{R}$ one lets

$$S(\alpha_0) := \sup_{\|\boldsymbol{\alpha}\|\leq\alpha_0} \|V_{\boldsymbol{\alpha}}\| \quad \text{for} \quad V_{\boldsymbol{\alpha}} := \mathrm{e}^{+\frac{1}{2}\boldsymbol{\alpha}.\boldsymbol{E}}V\mathrm{e}^{-\frac{1}{2}\boldsymbol{\alpha}.\boldsymbol{E}}. \tag{24}$$

In comparison with inequality (20) for the moment generating function of ϕ_0, the inequality (23) does not prove a uniform bound for the exponential moments of $\boldsymbol{\phi}$ (remark the t in the exponential). One can further prove, using a proof which is again conceptually elementary, but technically more delicate (Proposition 2.7 of [P21]) that if the initial state is the multi-thermal state $\rho_{\boldsymbol{\beta}}$, then

$$\chi_t(\boldsymbol{\alpha})\,\mathrm{e}^{-|\theta|\,3S_{\boldsymbol{\beta}}(\theta,\|\boldsymbol{\alpha}\|)} \leq \chi_t(\boldsymbol{\alpha}+\theta\mathbf{1}) \leq \chi_t(\boldsymbol{\alpha})\,\mathrm{e}^{+|\theta|\,3S_{\boldsymbol{\beta}}(\theta,\|\boldsymbol{\alpha}\|)} \tag{25}$$

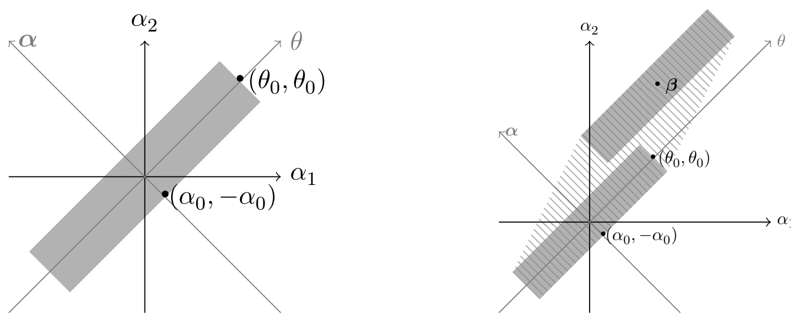
where for $\theta_0 \in \mathbb{R}$ and $\alpha_0 \in \mathbb{R}_+$,

$$S_{\boldsymbol{\beta}}(\alpha_0,\theta_0) = \sup_{|\theta|\leq|\theta_0|}\ \sup_{\boldsymbol{\alpha}\in\mathcal{B}(\alpha_0)} \|V_{\boldsymbol{\alpha}+\theta\mathbf{1}}\| + \sup_{|\theta|\leq|\theta_0|}\ \sup_{\boldsymbol{\alpha}\in\mathcal{B}(\alpha_0)} \|V_{\boldsymbol{\beta}+\boldsymbol{\alpha}+\theta\mathbf{1}}\| \tag{26}$$

where

$$\mathcal{B}(\alpha_0) := \{\boldsymbol{\alpha} \in \mathbb{R}^\ell : \boldsymbol{\alpha}.\mathbf{1} = 0, \|\boldsymbol{\alpha}\| < \alpha_0\}.$$

To be more picturesque, in the case $\ell = 2$, the quantities $S(\alpha_0, \theta_0)$ and $S_{\boldsymbol{\beta}}(\alpha_0, \theta_0)$ give uniform bounds on the deformations $V_{\boldsymbol{\alpha}}$ for $\boldsymbol{\alpha}$ in, respectively, the grey zone pictured on the left below, and the union of the two grey zones pictured on the right (and by convexity, in the hatched zone).



Assume once again that the system of interest is described by a sequence $(\mathcal{H}^{(N)}, \rho^{(N)}, \boldsymbol{E}^{(N)}, V^{(N)})_N$ of finite spaces , each equipped with a state, a ℓ-tuple of Hamiltonians $\boldsymbol{E} = (H_1^{(N)}, \dots, H_\ell^{(N)})$, and a coupling $V^{(N)}$. We denote once more with a subscript (N) all objects associated with the N-th finite system. We will assume in the sequel that for all $t \in \mathbb{R}$,

$$\chi_t(\boldsymbol{\alpha}) := \lim_{N\to\infty} \chi_t^{(N)}(\boldsymbol{\alpha}) \text{ exists for } \|\boldsymbol{\alpha}\| < \alpha_0. \tag{27}$$

In that case, there exists for all t a probability $\mathbb{P}_{\boldsymbol{E},t}$ on $\mathbb{R}$ which is the limit, in the sense of exponential moments, of the sequence of $\mathbb{P}^{(N)}_{\boldsymbol{E}^{(N)},t}$ associated with $(\mathcal{H}^{(N)}, \rho^{(N)}, \boldsymbol{E}^{(N)}, V^{(N)})$.

Remark 2.13. When the system is multi-thermal, one has $-\log\rho = \sum_i \beta_i H_i + \text{constant}$. The random variable ς of section 3 then reads $\varsigma = \sum_i \beta_i \phi_i$ and can then be expressed in the probability space of the present section. We will therefore use the same notation $\mathbb{P}_t$ for the probability measures of these two sections. In addition, with the notation of the previous section, $e_t(\alpha) = \chi_t(\alpha\boldsymbol{\beta})$, and under the assumption of time-reversal invariance, the symmetry $e_t(\alpha) = e_t(1-\alpha)$ has the following analogue:

$$\chi_t(\boldsymbol{\alpha}) = \chi_t(\boldsymbol{\beta} - \boldsymbol{\alpha}). \tag{28}$$

.

An immediat consequence of (25) is the following result:

Theorem 2.14 ([P21]). *Assume that* $\sup_N S_{\boldsymbol{\beta}}^{(N)}(\alpha_0, \theta_0) < \infty$. *Then the functional defined by* $\overline{\chi}_+(\boldsymbol{\alpha}) = \limsup_{t\to\infty} \frac{1}{t} \log \chi_t(\boldsymbol{\alpha})$ *satisfies*

$$\overline{\chi}_+(\boldsymbol{\alpha}) = \overline{\chi}_+(\boldsymbol{\alpha} + \theta\mathbf{1}) \tag{29}$$

for $\|\boldsymbol{\alpha}\| \le \alpha_0$ *and* $|\theta| \le \theta_0$. *In particular,* $\chi_+(\theta\mathbf{1}) = 0$ *for* $|\theta| \le \theta_0$.

Remark 2.15. The symmetry (29) was suggested by Andrieux, Gaspard, Monnai and Tasaki in [5] and we will therefore call it the AGMT symmetry. This article uses however the assumption $\sup_N S_{\boldsymbol{\beta}}^{(N)}(\alpha_0, \theta_0) < \infty$ but claims that it is a consequence of the uniform boundedness of couplings $\sup_N \|V^{(N)}\| < \infty$, and this is false in general.

Remark 2.16. The fluctuation relation (15) followed directly from the finite-time relation (12). The translation invariance (29), on the contrary, only holds in the limit $t \to \infty$.

For $\boldsymbol{s} \in \mathbb{R}^\ell$, we define

$$\bar{I}(\boldsymbol{s}) = \sup_{\boldsymbol{\alpha} \in \mathbb{R}^\ell} \big(\boldsymbol{\alpha}.\boldsymbol{s} - \overline{\chi}_+(\boldsymbol{\alpha})\big) \in [0, +\infty].$$

If $\sup_N S_{\boldsymbol{\beta}}^{(N)}(\alpha_0, \theta_0) < \infty$, we have immediately by (29)

$$\bar{I}(\boldsymbol{s}) \geq \theta_0 \sum_i s_i = \theta_0 \, |\boldsymbol{s}.\mathbf{1}|. \tag{30}$$

In particular, if for all $\theta_0 \geq 0$ there exists α_0 for which $S_{\boldsymbol{\beta}}(\alpha_0, \theta_0) < \infty$, then $\bar{I}(\boldsymbol{s}) = +\infty$ unless $\boldsymbol{s} \in \Sigma$. If $\sup_N S_{\boldsymbol{\beta}}^{(N)}(\alpha_0, \theta_0) < \infty$ and that the system is time-reversal invariant, then

$$\bar{I}(\boldsymbol{s}) = \bar{I}(-\boldsymbol{s}) - \boldsymbol{\beta}.\boldsymbol{s}. \tag{31}$$

Remark 2.17. One can also prove an AGMT relation

$$\overline{\chi}_+(\boldsymbol{\alpha}) = \overline{\chi}_+(\boldsymbol{\alpha} + \theta \mathbf{1})$$

for $\|\boldsymbol{\alpha}\| < \alpha_0$ and for all θ, in a system which is not necessarily multi-thermal, if $\sup_N S^{(N)}(\alpha_0, \theta_0) < \infty$ for some α_0 and for all θ_0, but we have no non-trivial example where such an assumption can be proven.

To state the other consequences of the invariance relation (29), from now on we implicitly suppose that the quantity $\chi_+(\boldsymbol{\alpha}) = \lim_{t\to\infty} \frac{1}{t} \log \chi_t(\boldsymbol{\alpha})$ exists whenever we invoke it. The Theorems 3.14, 3.15 and 3.16 of [P21] show:

Theorem 2.18 ([P21]). *Assume that the system is multi-thermal at inverse temperatures $\boldsymbol{\beta}$ and that the assumption $\sup S_\beta^{(N)}(\alpha_0, \theta_0) < \infty$ holds for non-zero α_0 and θ_0. Then:*

1. *If χ_+ is differentiable at 0, the random variable $\boldsymbol{\phi}$ converges exponentially to the vector $\langle \boldsymbol{\Phi} \rangle_+$ of $\mathbb{R}^\ell$ of coordinates*

   $$\langle \Phi_j \rangle_+ = -\frac{\partial \chi_+}{\partial \alpha_j}\Big|_{\alpha_j = 0}$$

   for $j = 1, \ldots, \ell$, which satisfies $\sum_j \langle \Phi_j \rangle_+ = 0$.

2. *If there exists a complex neighbourhood of the origin $\mathcal{V}$ such that $\sup_{t>1} \sup_{\boldsymbol{\alpha} \in \mathcal{V}} \frac{1}{t} |\log \chi_t(\boldsymbol{\alpha})| < \infty$, then we have convergence in distribution:*

   $$\sqrt{t}\big(\boldsymbol{\phi} - \langle \boldsymbol{\Phi} \rangle_+\big) \underset{t\to\infty}{\longrightarrow} \mathcal{N}(0, \boldsymbol{D}) \tag{32}$$

   where

   $$D_{i,j} = \frac{\partial^2 \chi_+}{\partial \alpha_i \partial \alpha_j}\Big|_{\boldsymbol{\alpha} = \mathbf{0}} = \lim_{t\to\infty} \frac{1}{t} \Big(\frac{\partial^2 \chi_t}{\partial \alpha_i \partial \alpha_j} - \frac{\partial \chi_t}{\partial \alpha_i} \frac{\partial \chi_t}{\partial \alpha_i} \Big)\Big|_{\boldsymbol{\alpha} = \mathbf{0}}. \tag{33}$$

   The covariance matrix $\boldsymbol{D}$ is degenerate since $\sum_{i,j} D_{i,j} = 0$ by (18).

3. *If $\chi_+(\boldsymbol{\alpha})$ exists in $[-\infty, +\infty]$ for all $\boldsymbol{\alpha} \in \mathbb{R}^\ell$, then for any Borel set $\boldsymbol{B}$ of $\mathbb{R}^\ell$,*

$$-\inf_{\boldsymbol{s}\in\mathrm{int}(\boldsymbol{B})\cap\mathcal{F}} I(\boldsymbol{s}) \leq \liminf_{t\to\infty} \frac{1}{t}\log \mathbb{P}_t(\boldsymbol{B}) \leq \limsup_{t\to\infty} \frac{1}{t}\log \mathbb{P}_t(\boldsymbol{B}) \leq -\inf_{\boldsymbol{s}\in\mathrm{fer}(\boldsymbol{B})} I(\boldsymbol{s}) \quad (34)$$

where $\mathcal{F}$, the set of exposed points[5] of I, is equal to $\mathbb{R}^\ell$ if χ_+ is assumed everywhere finite and differentiable. In addition, $I(\boldsymbol{s}) \geq \theta_0 |\boldsymbol{s}.\mathbf{1}|$ and if the system is time-reversal invariant, then also $I(\boldsymbol{s}) = I(-\boldsymbol{s}) - \boldsymbol{\beta}.\boldsymbol{s}$.

Remark 2.19. Once again, in many situations where the physical system of interest $\mathcal{O}$ can be approximated by confined systems $\mathcal{O}^{(N)}$, one can show not only that the associated functionals $e_t^{(N)}(\alpha)$ converge for all t and α of $\mathbb{R}$, but also that $\langle\Phi_j\rangle_+ = \lim_{t\to\infty}\frac{1}{t}\int_0^t \rho\circ\tau^s(\Phi_j)\,\mathrm{d}s$, which is equal to $\rho_+(\Phi_j)$ when the NESS ρ_+ exists and is unique, and

$$D_{i,j} = \lim_{t\to\infty}\frac{1}{t}\int_0^t\int_0^t \rho_{\boldsymbol{\beta}}\Big(\tau^{s_1}\big(\Phi_i - \rho_{\boldsymbol{\beta}}(\Phi_i)\big)\tau^{s_2}\big(\Phi_j - \rho_{\boldsymbol{\beta}}(\Phi_j)\big)\Big)\mathrm{d}s_1\mathrm{d}s_2,$$

where Φ_i is the observable for the flux coming out of $\mathcal{R}_i$, mentioned at the beginning of chapter 1. As a consequence, contrary to what we have observed in chapter 1, the definition of variations of heat by two-time measurements allows to describe the asymptotic distribution of the joint fluctuations of these variations even out of equilibrium. This asymptotic distribution is Gaussian, and its covariance can be expressed as a correlation which, in the equilibrium situations (i.e. when $\boldsymbol{\beta}$ is of the form $\boldsymbol{\beta}_{\mathrm{eq}} = (\beta_0, \ldots, \beta_0)$), are the Kubo transport coefficients (regarding this identity, see Proposition 5.8 of [P11]).

Let us now shortly discuss applications of the above results to explicit models: the uniform bound assumptions $\sup_N S^{(N)}(\alpha_0, \theta_0) < \infty$ or $\sup_N S_{\boldsymbol{\beta}}^{(N)}(\alpha_0, \theta_0) < \infty$ are in general ensured by "ultraviolet assumptions": for example, in a model of spin-fermion type as in section 1.3, the relevant assumptions will be that the form factors f_k and g_k appearing in the coupling term V are in the domain of $\mathrm{e}^{\alpha h_0}$ for α in a real neighbourhood of the origin. Under such an assumption, one can in general prove that $\sup_N S^{(N)}(\alpha_0, \theta_0) < \infty$ either for all α_0 and θ_0; or for α_0 and θ_0 small enough. In this latter case, one obtains $\sup_N S_{\boldsymbol{\beta}}^{(N)}(\alpha_0', \theta_0') < \infty$ for $\boldsymbol{\beta}$ sufficiently small (and for α_0', β_0' inferior to α_0, β_0), so that our result typically in a "high temperature" regime. Proving that χ_+ exists is a similar problem to the one we encountered in section 3 concerning e_+, and we refere the reader to the discussion at the end of that section.

Remark 2.20. Point 2. of Theorem 2.18 is precisely the type of result that chapter 1 was aiming for. It does describe a joint distribution, and raises no conceptual difficulty. In a model of spin-fermion type, one can apply both the results of chapter 1 and those of the present chapter; it then holds that in equilibrium situations $\boldsymbol{\beta} = (\beta_0, \ldots, \beta_0)$, both definitions of the variations of entropy (naive and through two-time measurements) have normal fluctuations gaussiennes, and then their covariances are equal (which was expected, due to relation (8)).

5. Application: linear response theory

It is well-known that, in the classical case, the fluctuation relations allow to recover linear response theory, and this allows to view them as the right extension of that theory in situations far from equilibrium.

[5] See Definition 2.3.3 of [45]

We are going to prove this result starting from two symmetries: the fluctuation relations (15) and the translation invariance symmetry (29). Assume therefore that we consider a system parameterized by $\boldsymbol{\beta} \in (\beta_0 - \delta, \beta_0 + \delta)^\ell$, satisfying (27) and $\sup_N S_{\boldsymbol{\beta}}^{(N)}(\alpha_0, \theta_0) < \infty$ for all those $\boldsymbol{\beta}$. If we assume that the corresponding quantities $\chi_+(\boldsymbol{\beta}, \boldsymbol{\alpha})$ exist and are C^1 in $\boldsymbol{\beta}$ and C^2 in $\boldsymbol{\alpha}$ in a neighbourhood of $\boldsymbol{\beta}_{\text{eq}} = (\beta_0, \dots, \beta_0)$ and of $(0, \dots, 0)$ respectively, then the transport coefficients can be written as

$$L_{i,j} = \frac{\partial \langle \Phi_j \rangle_{\boldsymbol{\beta}}}{\partial \beta_i}\Big|_{\boldsymbol{\beta}_{\text{eq}} = \mathbf{0}}.$$

The ES and AGMT symmetries, respectively (28) and (29), show that

$$\chi_+(\boldsymbol{\beta}, \boldsymbol{\alpha}) = \chi_+(\boldsymbol{\beta}, \boldsymbol{\alpha} + \boldsymbol{\beta}_{\text{eq}}) = \chi_+(\boldsymbol{\beta}_{\text{eq}} + \boldsymbol{\zeta}, (\boldsymbol{\beta} - \boldsymbol{\beta}_{\text{eq}}) - \boldsymbol{\alpha}). \tag{35}$$

Differentiating this identity twice, we obtain

$$-\frac{\partial^2 \chi_+(\boldsymbol{\beta}, \boldsymbol{\alpha})}{\partial \beta_j \partial \alpha_k}\Big|_{\boldsymbol{\alpha}=\mathbf{0}, \boldsymbol{\beta}=\boldsymbol{\beta}_{\text{eq}}} = \frac{1}{2} \frac{\partial^2 \chi_+(\boldsymbol{\beta}, \boldsymbol{\alpha})}{\partial \alpha_j \partial \alpha_k}\Big|_{\boldsymbol{\alpha}=\mathbf{0}, \boldsymbol{\beta}=\boldsymbol{\beta}_{\text{eq}}}.$$

But from (8), the first term is $L_{i,j}$ and from (33), the second one is $D_{i,j}$. We therefore have

$$2L_{i,j} = D_{i,j}(\boldsymbol{\beta}_{\text{eq}}),$$

and in particular $L_{i,j} = L_{j,i}$. We have therefore proven the Kubo formula and Onsager reciprocity relations.

Remark 2.21. We have used a symmetry $\chi_+(\boldsymbol{\beta}, \boldsymbol{\alpha}) = \chi_+(\boldsymbol{\beta}_{\text{eq}} + \boldsymbol{\zeta}, (\boldsymbol{\beta} - \boldsymbol{\beta}_{\text{eq}}) - \boldsymbol{\alpha})$ which is analogous to the "generalized Evans-Searles symmetry" (GES) of [P11]. The latter was however proven only for the functional e_∞, which had no operational interpretation (see section 2.4).

Remark 2.22. The two observed symmetries, the ES symmetry (15) and the AGMT Symmetry (29), have probably not been used to their full extent yet. As we wrote above, (15) follows from the finite-time symmetry (12) and the additional information given by e_+ in Proposition 2.6 proceeds exclusively from the assumption that e_+ exists, not of the symmetry which trivially survives the passage to infinite time. On the contraty, (29) is only true in infinite time but the relevant information in Theorem 2.18 essentially reduces to that of Theorem 2.11, which concerns the total energy and does not use (29). We have seen that (15) and (29) alone imply the Kubo formula and Onsager reciprocity relations, but this implication only uses a tiny part of the information they carry. We think therefore that it is possible to obtain more implications of these two symmetries, for example as contraints on the rate function of $\boldsymbol{E}$ that would incorporate both the first and the second laws.

Remark 2.23. We have been able to show in this chapter results concerning the fluctuations of the variations of entropy and of heat (either total or detailed for each reservoir), therefore giving in a quantum setup statistical formulations of the first and second principles that go beyond the properties of averages. One can then wonder if it is possible to apply the same program to further physical principles principes. A natural candidate was the Landauer principle; its investigation for the so-called repeated interaction systems is the subject of chapter 4.

Chapter 3

Repeated measurements of a system: the outcomes and the system

In this section, we will discuss the properties of the outcomes and of the post-measurement random state in the framework of repeated measurements. We will begin by giving more precise definitions for the formalism introduced in section 0.3. This formalism concerns discrete-time evolutions; continuous-time extensions will then be described in sections 4 and 5.

We consider a separable Hilbert space $\mathcal{H}$, and suppose that its initial state is represented by a density matrix ρ. We consider a family $(\Phi_i)_{i\in V}$ of completely positive maps (see annex A) such that $\Phi := \sum_{i\in V}\Phi_i$ preserves the trace. Such a family is called an *instrument* (see [44,76]), and a completely positive trace-preserving map like Φ is called *quantum channel*. We define a probability on V^n by

$$\mathbb{P}_\rho(i_1,\ldots,i_n) = \mathrm{tr}\left(\Phi_{i_n}\circ\ldots\circ\Phi_{i_1}(\rho)\right). \tag{1}$$

The fact that $\Phi^*(\mathrm{Id}) = \mathrm{Id}$ shows that this definition is consistent with the definition on V^{n+1}, so that (by Kolmogorov's extension Theorem), the prescription (1) defines a probability measure on $V^{\mathbb{N}}$. We then define two random variables on $\Omega = V^{\mathbb{N}}$ equipped with its product σ-algebra:

$$m_n(\omega) = i_n \qquad \rho_n(\omega) = \frac{\Phi_{i_n}\circ\ldots\circ\Phi_{i_1}(\rho)}{\mathrm{tr}\left(\Phi_{i_n}\circ\ldots\circ\Phi_{i_1}(\rho)\right)} \tag{2}$$

if $\omega = (i_1, i_2, \ldots)$. The sequence $(m_n)_n$ is not in general a Markov chain, but the sequence $(\rho_n)_n$ is. We have the immediate identity

$$\mathbb{E}_\rho(\rho_n) = \Phi^n(\rho). \tag{3}$$

The process $(\rho_n)_n$ is called a *quantum trajectory*; quantum trajectories were introduced (with a continuous-time parameter, as discussed in sections 4 and 5) as a model for the "wave packet projection" (1) by Diosi and Gisin in [49] and [65] respectively, and as a tool for numerical simulation in [40] by Dalibard, Castin and Mølmer.

Remark 3.1. Two remarkable properties are immediate:

- if $\Phi(\rho) = \rho$, then $\mathbb{P}_\rho$ is invariant under the left shift,
- if in addition this ρ is the unique invariant state of Φ, then $\mathbb{P}_\rho$ is ergodic.

These two observations are due to Fannes, Nachtergaele and Werner in [59], and allow one to exploit the theory of ergodic processes, if one assumes that the initial state ρ is Φ-invariant.

There are essentially two fundamental results on repeated measurements, and both are due to Kümmerer and Maassen. The first is the ergodic theorem for states which originates in [87], but which we here give in the infinite-dimensional version suggested by Lim in [93].

Theorem 3.2. *Assume that the channel Φ admits a unique invariant state ρ_{inv}. Then for any initial state ρ, one has $\mathbb{P}_\rho$-almost surely the weak convergence*

$$\frac{1}{n}\sum_{k=0}^{n-1}\rho_k \to \rho_{\mathrm{inv}}$$

(i.e. one has convergence of the above functionals when evaluated at any $X \in \mathcal{B}(\mathcal{H})$).

The second fundamental result is the purification Theorem (see [95]). It concerns instruments which are perfect, in the sense that every Φ_i is of the form $\Phi_i(\eta) = L_i \eta L_i^*$, as is the case when the indirect measurement is a two-time measurement and concerns a non-degenerate observable, see example 3.36. In that case, it is clear that if ρ_n is pure, then ρ_{n+1} will be pure; more precisely, if $\rho_n = |x_n\rangle\langle x_n|$, then $\rho_{n+1} = |x_{n+1}\rangle\langle x_{n+1}|$, where

$$\mathbb{P}\big(x_{n+1} = \tfrac{L_i x_n}{\|L_i x_n\|} \mid x_n\big) = \|L_i x_n\|^2. \tag{4}$$

As a consequence, the evolution preserves the purity of states. The purification Theorem of Kümmerer and Maassen gives a condition under which ρ_n is asymptotically pure regardless of ρ:

Theorem 3.3. *Assume that the considered instrument $(\Phi_i)_{i\in V}$ is perfect and associated with operators $(L_i)_{i\in I}$ that have the property that any orthogonal projector π such that $\pi L_{i_1}^* \dots L_{i_n}^* L_{i_n} \dots L_{i_1}\pi \propto \pi$ for any $(i_1,\dots,i_n)$ and any n of I and $\mathbb{N}^*$ is of rank 0 or 1. Then for any initial state ρ, one has $\mathbb{P}_\rho$-almost surely $\mathrm{tr}(\rho_n^2) \underset{n\to\infty}{\longrightarrow} 1$.*

Here $X \propto Y$ means that there exists $\lambda \in \mathbb{C}$ such that $X = \lambda Y$ or $\lambda X = Y$. Since the only states η satisfying $\mathrm{tr}(\eta^2) = 1$ are the pure states, $\lim_{n\to\infty} \mathrm{tr}(\rho_n^2) = 1$ means that the sequence $(\rho_n)_n$ is asymptotically pure. One can see that the assumption of Theorem 3.3 means that there exists no subspace of dimension higher than 2, on which all the $L_{i_n}\dots L_{i_1}$ are proportional to unitaries. Without this assumption, an initial state ρ of rank at least 2, but with support in such a subspace will induce a sequence $(\rho_n)_n$ of constant rank, so that this assumption is necessary for purification. The same remark shows that if $\dim\mathcal{H} = 2$ then the assumption of Theorem 3.3 holds unless all the L_i are proportional to unitary operators.

1. Repeated measurements of a system: the outcomes

In this section, we generalize the framework of generalized measurements described above, allowing the instrument $(\Phi_i)_{i\in V}$ considered at time $n+1$ to depend on the previous measurements $m_1,\dots,m_n$. This will allow to model the situations where, for example, the parameters ξ, U, M of the indirect measurement decribed in section 0.3 are chosen depending on earlier measurements. This should in particular be useful to study the control of states of the system $\mathcal{S}$ by retroaction from indirect measurements. In order to achieve this description, we will include the measurement outcomes in the state space, and suppose that the considered Hilbert space can be written as $\mathcal{H} = \bigoplus_{i\in V}\mathfrak{h}_i$ where V is a discrete ensemble representing the information recorded from earlier measurements (for example, $V = \mathrm{sp}\, M$ if one only keeps track of the last measurement, or $V = \mathbb{N}^{\mathrm{sp}\,M}$ if one keeps the full record of counts for each possible outcome value).

We will simplify the notation by writing as $x \otimes |i\rangle$ any vector $x \in \mathfrak{h}_i$, so that $\mathcal{H}$ is $\bigoplus \mathfrak{h}_i \otimes |i\rangle$. One then interprets the coordinate i as the position of a particle on the set V, and the coordinate in $\mathfrak{h}_i$ as an internal degree of freedom of the particle when it sits at i. We then consider a family $(\phi_{i,j})_{i,j\in V}$ of completely positive maps from $\mathcal{I}_1(\mathfrak{h}_j)$ to $\mathcal{I}_1(\mathfrak{h}_i)$ such that $\sum_{i\in V} \mathrm{tr}_{\mathfrak{h}_i}\left(\phi_{i,j}(\eta)\right) = \mathrm{tr}_{\mathfrak{h}_j}(\eta)$ for all $\eta \in \mathcal{I}_1(\mathfrak{h}_j)$. One can then define an instrument $\left(\Phi_{i,j}\right)_{i,j\in V}$ by

$$\Phi_{i,j}\Big(\sum_{k,l} \rho_{k,l} \otimes |k\rangle\langle l|\Big) = \phi_{i,j}(\rho_{j,j}) \otimes |i\rangle\langle i|. \tag{5}$$

The completely positive map $\Phi_{i,j}$ then encodes both the probability of the transition $j \rightsquigarrow i$, and the effect on the internal degrees of freedom.

Because expression (5) only depends on the block-diagonal terms $\rho_{j,j}$ and outputs a block-diagonal operator, we assume from now on that all states are of the form $\rho = \sum_{i\in V} \rho_i \otimes |i\rangle\langle i|$ (it will anyway be the case after the application of one $\Phi_{i,j}$); otherwise, it suffices to replace in all expressions ρ by its block-diagonal restriction .

We define as in (1) a probability on $\Omega = V^{\mathbb{N}}$ by

$$\mathbb{P}_\rho(i_1,\ldots,i_n) = \mathrm{tr}\left(\phi_{i_n,i_{n-1}} \circ \ldots \circ \phi_{i_2,i_1}(\rho_{i_1})\right) \tag{6}$$

and as in (2) the random variables

$$v_n(\omega) = i_n \qquad \varrho_n(\omega) = \frac{\phi_{i_n,i_{n-1}} \circ \ldots \circ \phi_{i_2,i_1}(\rho_{i_1})}{\mathrm{tr}\left(\phi_{i_n,i_{n-1}} \circ \ldots \circ \phi_{i_2,i_1}(\rho_{i_1})\right)} \tag{7}$$

if $\omega = (i_1, i_2, \ldots)$. Note that the ρ_n of (2) would be $\varrho_n \otimes |v_n\rangle\langle v_n|$: we use a different notation to emphasize that ϱ_n is a state of $\mathfrak{h}_{v_n}$ and not of $\mathcal{H}$. Now, neither $(v_n)_n$, nor $(\varrho_n)_n$ are Markov chains in general, but the sequence of pairs $(v_n, \varrho_n)_n$ is one.

Remark 3.4. It is clear that this model extends that of classical Markov chains: precisely, if one has a transition matrix $\Pi = (\Pi_{i,j})_{i,j\in V}$ (where $\Pi_{i,j}$ is the probability to go to j when sitting at i), then with $\mathfrak{h}_i \equiv \mathbb{C}$ and $\phi_{i,j} = \Pi_{j,i}$, any diagonal state $\sum_{i\in V} p_i \otimes |i\rangle\langle i|$ is mapped to a state $\sum_{i\in V} q_i \otimes |i\rangle\langle i|$, with $q = p\Pi$. We will call this a "minimal dilation" of the Markov chain.

The case where each $\phi_{i,j}$ is pure, i.e. of the form $\rho \mapsto L_{i,j}\rho L_{i,j}^*$ (where $L_{i,j}$ is an operator from $\mathfrak{h}_j$ to $\mathfrak{h}_i$) is called "open quantum walk" (OQW). The study of open quantum walks started with [7] where these walks are presented as a new quantum version of Markov chains, and has given rise to a numerous literature. For this reason, the articles [P13, P15, P17] are written with pure $\phi_{i,j}$, and the case of general $\phi_{i,j}$ is called "extended open quantum walk". All results from these articles extend immediately to the case of non-pure transitions $\phi_{i,j}$.

Remark 3.5. In spite of their name, open quantum walks are very different from the more common unitary quantum walks, initially studied in the framework of quantum algorithms (see [84]): in particular, the fact that one measures the "position" at each step destroys interferences between the different possible trajectories (contrarily to the case of unitary walks), see [P23].

Remark 3.6. Beyond their use for the control of quantum systems, the most natural application of open quantum walks (extended, and in continuous time – see section 4) may come from the fact that they can be obtained by a weak coupling limit (*à la* Davies [43]) starting from a system with a degenerate Hamiltonian (once more, see [P23]).

The main motivation behind articles [P13, P15, P17] was to study the properties of open quantum walks by analogy with Markov chains. As in the classical case, the ergodicity properties of the probability $\mathbb{P}_\rho$ are related to those of the map Φ, that depend in particular on its peripheral spectrum. The basic tool for the study of the latter is the "Perron–Frobenius Theorem" for quantum channels, originally proved in [56] and extended (partly) to the infinite-dimensional case [53, 68]. We recall all these results in annex A; they essentially depend on an irreducibility property of Φ. A quantum channel on $\mathcal{H}$ is called *irreducible* (in the sense of Davies, [42]) if there exists no non-trivial orthogonal projector π of $\mathcal{H}$ such that $\Phi\big(\pi\mathcal{I}_1(\mathcal{H})\pi\big) \subset \pi\mathcal{I}_1(\mathcal{H})\pi$. It is easy to prove that if a quantum channel has a unique invariant state, and that the latter is faitfhul, then the channel is irreducible; the annex A recalls that if the channel is irreducible, then 1 is an eigenvalue of multiplicity at most 1, and that when 1 is an eigenvalue there is an associated strictly positive eigenvector. In particular, if a channel is irreducible then it admits at most one invariant state, which is then faitfhul.

An application of the Kümmerer–Maassen Theorem then gives the following results: if $\Phi = \sum_{i,j} \Phi_{i,j}$ is irreducible and admits an invariant state ρ_{inv} (which will then automatically be faitfhul), then the latter is of the form $\rho_{\text{inv}} = \sum_{i\in V} \rho_{\text{inv}}(i) \otimes |i\rangle\langle i|$, and for any initial state ρ, all i of V, the empirical frequency at i, defined by $N_{i,n} := \operatorname{card}\{k \le n \,|\, v_k = i\}$, satisfies

$$\begin{aligned} \frac{N_{i,n}}{n} &\underset{n\to\infty}{\longrightarrow} \operatorname{tr}\big(\rho_{\text{inv}}(i)\big) \quad \mathbb{P}_\rho\text{-almost surely,} \\ \frac{1}{N_{i,n}} \sum_{k=0}^{n-1} \rho_k \, \mathbb{1}_{v_k=i} &\underset{n\to\infty}{\longrightarrow} \frac{\rho_{\text{inv}}(i)}{\operatorname{tr}\big(\rho_{\text{inv}}(i)\big)} \quad \mathbb{P}_\rho\text{-almost surely.} \end{aligned} \tag{8}$$

The article [P15] characterises "in terms of paths" the irreducibility of Φ. For this, one defines for i, j of V a path from i to j as a sequence $i_0, \ldots, i_\ell$ in V where $\ell \ge 1$, such that $i_0 = i$ and $i_\ell = j$; such a path is said to be of length ℓ. We denote $\mathcal{P}(i,j)$ (respectively $\mathcal{P}_\ell(i,j)$) the set of paths from i to j of any length (respectively of length ℓ). A path from i to i is called a loop. For $\pi = (i_0, \ldots, i_\ell)$ in $\mathcal{P}(i,j)$ we denote by L_π the operator from $\mathfrak{h}_i$ to $\mathfrak{h}_j$ defined by

$$L_\pi = L_{i_\ell, i_{\ell-1}} \ldots L_{i_1, i_0} = L_{j, i_{\ell-1}} \ldots L_{i_1, i}. \tag{9}$$

We then have:

Proposition 3.7 ([P15]). *The quantum channel Φ is irreducible if and only if, for all i and j of V, one has one of the following equivalent conditions:*

- *for all x of $\mathfrak{h}_i \setminus \{0\}$, the set $\{L_\pi x \,|\, \pi \in \mathcal{P}(i,j)\}$ is total in $\mathfrak{h}_j$,*
- *for all x of $\mathfrak{h}_i \setminus \{0\}$ and y of $\mathfrak{h}_j \setminus \{0\}$, there exists a path π of $\mathcal{P}(i,j)$ such that $\langle y, L_\pi x\rangle \neq 0$.*

Remark 3.8. A Markov chain is irreducible if, on the graph induced by the transitions of non-zero probability, any point is accessible from any point. Proposition 3.7 extends this result by including the notion of internal degrees of freedom to the accessibility criterion. The latter reduces to the usual criterion if the considered open quantum walk is the minimal dilation of a Markov chain (this is not true if one considers non-minimal dilatations as in [7]). Remark however that the irreducibility of Φ is not the same as the irreducibility of $(v_n)_n$ as a Markov chain, and that this last notion is actually ill-defined, see remark 3.41.

As "picturesque" as it may be, the irreducibility criterion given by Proposition 3.7 is not in general easy to apply: see example 3.9 below.

Example 3.9. Consider the open quantum walk on $V = \{1, \dots, d\}$ (respectively $V = \mathbb{Z}$) and $\mathfrak{h}_i \equiv \mathbb{C}^2$ for all i, with $L_{i,i+1} = L_-$, $L_{i+1,i} = L_+$ for all i (where $i+1$ is understood *modulo* d if $V = \{1, \dots, d\}$), and L_-, L_+ are two operators on $\mathfrak{h}$ that satisfy $L_-^* L_- + L_+^* L_+ = \mathrm{Id}_{\mathbb{C}^2}$. We call this the simple open quantum walk on V. One can then show by a tedious proof that

- the simple OQW on $\{1, \dots, d\}$ is irreducible if and only if the set of L_π, $\pi \in \mathcal{P}(0,0)$, has no eigenvector in common;
- the simple OQW on $\mathbb{Z}$ is irreducible unless there exists an eigenvector common to L_+L_-, L_-L_+, and L_-; or an eigenvector common to L_+L_-, L_-L_+, and L_+ ; or there exists two linearly independent vectors e_0 and e_1 such that $L_+e_0, L_-e_0 \in \mathbb{C}e_1$ and $L_+e_1, L_-e_1 \in \mathbb{C}e_0$.

Remark 3.10. Proposition 3.7 can be adapted to the case of an extended OQW, i.e. to the case where the transitions from j to i are represented by a completely positive map $\Phi_{i,j}$. Indeed, this $\Phi_{i,j}$ admits a so-called Kraus decomposition $\Phi_{i,j}(\eta) = \sum_{K_i} L_{i,j}^{(k)} \eta L_{i,j}^{(k)\,*}$ (see for example [110]). Proposition 3.7 remains true if the path π is no longer chosen on the graph of vertices V but on the multigraph where edges from j to i correspond to the different $L_{i,j}^{(k)}$, $k \in K_i$, the operator L_π being defined consistently as a product of operators $L_{i,j}^{(k)}$.

The Perron–Frobenius Theorem allowed us to make sure that 1 is an eigenvalue of multiplicity at most 1; as for the transition matrices, the notion of period allows to make the form of the peripheral spectrum more precise, and, in the aperiodic case with $\dim \mathcal{H} < \infty$, to show that Φ^n converges as $n \to \infty$. The notion of period given in [56, 68] reduces in the case of open quantum walks to the following definition (which once again extends the classical definition):

Definition 3.11 ([P15]). *The period of Φ is the largest $d \in \mathbb{N}$ for which there exists for all $i \in V$ a resolution of identity*[1] $p_i(0), \dots p_i(d-1)$ *of $\mathfrak{h}_i$ such that for all $k = 0, \dots, d-1$, one has*

$$p_i(k) L_{i,j} = L_{i,j} p_j(k-1) \tag{10}$$

($k-1$ is to be understood modulo *d).*

Aperiodicity allows us to obtain a convergence in distribution in the case $\dim \mathcal{H} < \infty$:

Proposition 3.12 ([P15]). *If the open quantum walk is irreducible and aperiodic on $\mathcal{H}$ of finite dimension, then for any initial state ρ and any i of V,*

$$\mathbb{P}(v_n = i) \underset{n\to\infty}{\longrightarrow} \operatorname{tr}\big(\rho_{\mathrm{inv}}(i)\big).$$

In general, determining the period of an open quantum walk is difficult. As in the classical case, one can however obtain a simple sufficient condition for aperiodicity:

Proposition 3.13 ([P15]). *If there exists i of V such that for all x of $\mathfrak{h}_i$,*

$$\gcd\{\ell \geq 1, \exists \pi \in \mathcal{P}_\ell(i,i) \text{ such that } \langle x, L_\pi x\rangle \neq 0\} = 1,$$

then the open quantum walk is aperiodic.

Example 3.14. Excluding the trivial cases where $L_+ = 0$ or $L_- = 0$:

[1] A resolution of identity is a family of orthogonal projectors that sum up to identity.

- the d-simple OQW on $\{1,\ldots,d\}$ with odd d has period 1;
- the simple OQW on $\{1,\ldots,d\}$ with even d, or on $\mathbb{Z}$, has period 2 or 4.

In the second case, the period is 4 only if there exists an orthonormal basis of $\mathbb{C}^2$ in which L_+ is diagonal and L_- antidiagonal (or the other way around).

Remark 3.15. Once again, Definition 3.11 and Proposition 3.13 remain true for extended open quantum walks, if one considers paths on a multigraph.

The irreducibility assumption and the Perron–Frobenius Theorem allow us to prove easily a law of large numbers, a central limit Theorem and a large deviations principle in one blow. We will give the proof in the case of an OQW on a finite graph V, then state a similar result in the case of an homogeneous OQW on a lattice. Assume then that $V=\{1,\ldots,d\}$ and denote by N_n the d-tuple $\big(N_{1,n},\ldots,N_{d,n}\big)$. Then for all $\alpha\in\mathbb{R}^d$ one has

$$\begin{aligned}\mathbb{E}_\rho\big(\exp(\langle\alpha,N_n\rangle_{\mathbb{C}^v})\big)&=\sum_{i_1,\ldots,i_n}\exp\big((\alpha_{i_1}+\ldots+\alpha_{i_n})\big)\operatorname{tr}\big(\phi_{i_n,i_{n-1}}\circ\ldots\circ\phi_{i_2,i_1}(\rho_{i_1})\big)\\&=\operatorname{tr}\big(\Phi^{(\alpha)}\circ\ldots\circ\Phi^{(\alpha)}(\mathrm{e}^{\alpha_1}\rho_{i_1}\otimes|i_1\rangle\langle i_1|)\big)\end{aligned}$$

where
$$\Phi^{(\alpha)}=\sum_{i,j\in V}\mathrm{e}^{\alpha_i}\Phi_{i,j}.$$

By the criterion in Proposition 3.7, the operator $\Phi^{(\alpha)}$ is irreducible if Φ is, and therefore its spectral radius $\lambda^{(\alpha)}$ is a simple eigenvalue, with a definite-positive associated eigenvector. One can then show, using the periodic structure in Definition 3.11 (or Proposition 4.9, see remark 4.11) that

$$\frac{1}{n}\log\mathbb{E}_\rho\big(\exp(\langle\alpha,N_n\rangle_{\mathbb{C}^d})\big)\underset{n\to\infty}{\to}\log\lambda^{(\alpha)}.\tag{11}$$

In addition, by the perturbation theory of eigenvalues in finite dimension (see the first chapter of [83]), $\lambda^{(\alpha)}$ is locally analytic in α in the neighbourhood of any $\alpha\in\mathbb{R}$. As a consequence, $\alpha\mapsto\lambda^{(\alpha)}$ is infinitely differentiable on $\mathbb{R}$. We therefore obtain immediately a law of large numbers and a large deviations principle for $(v_n)_n$. In addition, an extra effort allows to show that the left-most term in (11) is bounded uniformly in n for u in a complex neighbourhood of the origin. One can then apply a multidimensional version of Bryc's Theorem (see annex A.4 of [P11]) to derive a central limit Theorem. The limiting mean and variance can be expressed as a function of λ, and from there one obtains more explicit expressions. One can then easily extend this result to the case where Φ has a unique invariant state which is not faithful. In the end one obtains the following result:

Proposition 3.16 ([P13]). *Let Φ be an open quantum walk on $V=\{1,\ldots,d\}$, that admits an invariant state ρ_{inv}. Define $m=(m_1,\ldots,m_d)\in\mathbb{R}^v$ and C a $d\times d$ symmetric real matrix by*

$$m_i=\operatorname{tr}\big(\rho_{\mathrm{inv}}(i)\big),\qquad\langle u,Cu\rangle=\lambda''_u-(\lambda'_u)^2$$

where λ'_u and λ''_u are the Gâteaux derivatives at 0, in the direction u, of λ. One then has

$$\frac{N_n}{n}\underset{n\to\infty}{\to}m\quad\mathbb{P}_\rho\text{-almost surely},\qquad\frac{N_n-m}{\sqrt{n}}\underset{n\to\infty}{\to}\mathcal{N}(0,C)\quad\text{in distribution,}$$

and N_n satisfies a large deviations principle for the good rate function

$$I(\nu) = \sup_{\alpha\in\mathbb{R}^v} \big(\langle\alpha,\nu\rangle - \lambda^{(\alpha)}\big).$$

One can give a similar statement for the open quantum walk such that:

- V is a lattice positively generated by a family S of p vectors $s_1,\ldots,s_p$,
- $\mathfrak{h}_i \equiv \mathfrak{h}$ for $\mathfrak{h}$ of fixed finite dimension,
- the walk is homogeneous in the sense that $\Phi_{i,j}$ is equal to a certain Φ_s if $i-j=s$ for some $s\in S$, and is otherwise null.

In that case one can define an "auxiliary" quantum channel on $\mathfrak{h}$ by $\psi = \sum_{s\in S}\phi_s$. A proof identical to that of Theorem 3.16 shows that

Proposition 3.17 ([P17]). *Assume that in the case described above, ψ has a unique invariant state η_{inv}. Define a vector m of $\mathbb{R}^d$ and a symmetric $d\times d$ real matrix C by*

$$m = \sum_{s\in S} s\,\mathrm{tr}\,\big(\phi_s(\eta_{\mathrm{inv}})\big), \qquad \langle u, Cu\rangle = \lambda''_u - (\lambda'_u)^2.$$

We then have

$$\frac{v_n}{n} \underset{n\to\infty}{\to} m \quad \mathbb{P}_\rho\text{-almost surely}, \qquad \frac{v_n - m}{\sqrt{n}} \underset{n\to\infty}{\to} \mathcal{N}(0,C) \quad \text{in distribution},$$

and $(N_n)_n$ satisfies a large deviations principle for the good rate function

$$I(\nu) = \sup_{\alpha\in\mathbb{R}^d} \big(\langle\alpha,\nu\rangle - \lambda^{(\alpha)}\big).$$

The proof is essentially identical to the one we have given for Theorem 3.16.

Remark 3.18. A central limit Theorem for $(v_n)_n$ as above had already been given when $V=\mathbb{Z}^d$ in [8]. The expression for the covariance in [8] is a priori different, but remark 5.15 of [P13] shows that both expressions actually coincide.

Remark 3.19. When the OQW is the minimal dilation of a classical Markov chain, the map ψ is trivial (it is the operator of multiplication by the constant 1) but the map $\alpha\mapsto\psi^{(\alpha)}$ is not. The above technique of proof is actually standard in the classical case (see section 3.1 of [45]).

Remark 3.20. The results concerning decompositions of a general Φ into irreducible OQW, described in section 6 of [P15] (and consequently expanded in [P14]) allow to extend these results to the case where the relevant quantum channels have more than one invariant state.

Let us pursue the study of open quantum walks in analogy with that of Markov chains: one of the first considered questions in a textbook treatment of the latter regards recurrence. Assume therefore temporarily that $(v_n)_n$ is a classical Markov chain on a discrete set V. For all i of V, we define

$$T_i = \inf\{n\geq 1\,|\, v_n = i\}, \qquad N_i = \mathrm{card}\{n\geq 1\,|\, v_n = i\}.$$

The classical results (see for example [51] or [104]) concerning return or passage times $(T_i)_{i\in V}$, and number of visits $(N_i)_{i\in V}$ show that for all i of V,

$$\mathbb{P}_i(T_i < \infty) = 1 \Leftrightarrow \mathbb{E}_i(N_i) = \infty. \tag{12}$$

This equivalence allows to define a notion of recurrence starting from either quantities $\mathbb{P}_i(T_i < \infty)$ or $\mathbb{E}_i(N_i)$. In addition, if the Markov chain is irreducible,

$$\mathbb{P}_i(T_i < \infty) < 1 \text{ for all } i \in V, \text{ or } \mathbb{P}_i(T_i < \infty) = 1 \text{ for all } i \in V, \tag{13}$$
$$\mathbb{E}_i(N_i) < \infty \text{ for all } i \in V, \text{ or } \mathbb{E}_i(N_i) = \infty \text{ for all } i \in V. \tag{14}$$

Similarly, for an irreducible Markov chain,

$$\mathbb{E}_i(T_i) < \infty \text{ for all } i \in V, \text{ or } \mathbb{E}_i(T_i) = \infty \text{ for all } i \in V. \tag{15}$$

If in addition the Markov chain admits an invariant probability measure $(\rho_i)_{i\in V}$, then

$$\mathbb{E}_i(T_i) = \rho_i^{-1} < \infty \text{ for all } i \in V. \tag{16}$$

The article [P17] studies in particular the analogues of properties (12) to (16) for open quantum walks. We will see that these properties are not all verified. To state our results, let us denote by $\mathbb{P}_{i,\varrho}$ the probability $\mathbb{P}_\rho$ in the case where $\rho = \varrho \otimes |i\rangle\langle i|$ with ϱ a state on $\mathfrak{h}_i$ (and similarly for $\mathbb{E}_{i,\varrho}$).

One can give results for general OQW (see [P17]), but one obtains clear-cut characterizations when one makes an assumption ensuring that the memory contained in the internal degrees of freedom is in a sense limited. We will therefore say that an open quantum walk on $\mathcal{H} = \bigoplus_{i\in V} \mathfrak{h}_i$ is *semifinite* if $\dim \mathfrak{h}_i < \infty$ for all $i \in V$.

The first result concerns analogues of (13) and (14) :

Theorem 3.21 ([P17]). *Let Φ be an irreducible semifinite open quantum walk. We are then in one (and only one) of the following situations:*

1. *for all i, j of V and ϱ of $\mathcal{S}(\mathfrak{h}_i)$, one has $\mathbb{E}_{i,\varrho}(N_j) = \infty$ and $\mathbb{P}_{i,\varrho}(T_j < \infty) = 1$;*
2. *for all i, j of V and ϱ of $\mathcal{S}(\mathfrak{h}_i)$, one has $\mathbb{E}_{i,\varrho}(N_j) < \infty$ and $\mathbb{P}_{i,\varrho}(T_i < \infty) < 1$;*
3. *for all i, j of V and ϱ of $\mathcal{S}(\mathfrak{h}_i)$, one has $\mathbb{E}_{i,\varrho}(N_j) < \infty$ but there exists i in V, ϱ, ϱ' in $\mathcal{S}(\mathfrak{h}_i)$ (ϱ necessarily non faitfhul) such that $\mathbb{P}_{i,\varrho}(T_i < \infty) = 1$ and $\mathbb{P}_{i,\varrho'}(T_i < \infty) < 1$.*

Remark 3.22. Remark 3.2 of [P17] gives examples of the three situations described above, and therefore shows that (12) and (13) are not satisfied for open quantum walks. The situation *3.* is obviously specific to the non-commutative case.

The proof of Theorem 3.21 uses three different arguments. First, one shows that one can construct for all i, j of V a completely positive contraction $\mathfrak{P}_{i,j}$ from $\mathcal{I}_1(\mathfrak{h}_j)$ to $\mathcal{I}_1(\mathfrak{h}_i)$ such that

$$\mathbb{P}_{i,\varrho}(T_j < \infty) = \operatorname{tr}\big(\mathfrak{P}_{j,i}(\varrho)\big), \qquad \mathbb{E}_{i,\varrho}(N_j) = \sum_{k\geq 0} \operatorname{tr}\big(\mathfrak{P}_{j,j}^k \circ \mathfrak{P}_{j,i}(\varrho)\big), \tag{17}$$

then that $\mathbb{P}_{i,\varrho}(t_j < \infty) = 1$ if and only if $\mathfrak{P}^*_{i,j}(\mathrm{Id}_{\mathfrak{h}_i})$ is of the form $\begin{pmatrix} \mathrm{Id} & 0 \\ 0 & * \end{pmatrix}$ in the decomposition $\mathfrak{h}_i = \mathrm{Ran}\,\varrho \oplus (\mathrm{Ran}\,\varrho)^{\perp}$, which explains the importance of faithful ϱ. Besides, one can see that for all j of V, the vector space

$$D^N(j) = \big\{x = \sum_{i \in V} v_i \otimes |i\rangle \text{ such that } \sum_{i\in V} \sum_{\pi \in \mathcal{P}(i,j)} \|L_\pi \varphi_i\|^2 < \infty\big\} \tag{18}$$

is stable by all $L_{k,l} \otimes |k\rangle\langle l|$ for k, l in V. This means that the orthogonal projector π on the closure of $D^N(j)$ satisfies $\Phi\big(\pi\mathcal{I}_1(\mathcal{H})\pi\big) \subset \pi\mathcal{I}_1(\mathcal{H})\pi$. If Φ is irreducible, then $\pi = 0$ or $\mathrm{Id}_{\mathcal{H}}$; yet, if $v_i \in D^N(j) \cap \mathfrak{h}_i$, one has $\mathbb{E}_{i,|v_i\rangle\langle v_i|}(N_j) < \infty$ and this allows to conclude (note that there is no similar argument concerning the probability $\mathbb{P}_{i,\varrho}(t_j < \infty)$). The third argument consists in showing that if $\dim \mathfrak{h}_i < \infty$ and that Φ is irreducible, then for all j of V one has $\inf_{\varrho \in \mathcal{S}(\mathfrak{h}_i)} \mathbb{P}_{i,\varrho}(t_j < \infty) > 0$, which allows to use arguments "à la" Markov to show that $\mathbb{E}_{i,\varrho}(N_j) = \infty$ implies $\mathbb{E}_{i,\varrho}(N_{j'}) = \infty$ for all j'.

One has on the other hand a result analogue to (15) for open quantum walks:

Theorem 3.23 ([P17]). *Let Φ be an irreducible semifinite open quantum walk. We are then in one (and only one) of the following situations:*

1. *for all i of V and ϱ of $\mathcal{S}(\mathfrak{h}_i)$, one has $\mathbb{E}_{i,\varrho}(T_i) < \infty$,*
2. *for all i of V and ϱ of $\mathcal{S}(\mathfrak{h}_i)$, one has $\mathbb{E}_{i,\varrho}(T_i) = \infty$.*

This result can be proven remarking that

$$D^T(j) = \big\{\varphi = \sum_{i \in V} \varphi_i \otimes |i\rangle \text{ such that } \sum_{i\in V} \sum_{\pi \in \mathcal{P}^{V\setminus\{j\}}(i,j)} \ell(\pi)\, \|L_\pi \varphi_i\|^2 < \infty\big\}$$

is stable by all $L_{k,l} \otimes |k\rangle\langle l|$ for k, l in V and using an irreducibility argument.

Last, one can establish a result concerning return times analogue to (16). To state it, let us define inductively for $j \in V$ a time of k-th return for $k \in \mathbb{N}$:

$$T_j^{(k)} = \inf\{n > T_j^{(k-1)} \,|\, v_n = j\}.$$

One then has:

Theorem 3.24 ([P17]). *Let Φ be an irreducible and semifinite open quantum walk, admitting an invariant state $\rho_{\mathrm{inv}} = \sum_{i \in V} \rho_{\mathrm{inv}}(i) \otimes |i\rangle\langle i|$. Then this walk is in situation 1 of Theorem 3.23, and for all i, j of V and η in $\mathcal{S}(\mathfrak{h}_i)$, the sequence $(T_j^{(k)}/k)_k$ converges $\mathbb{P}_{i,\rho}$-almost surely and in $\mathrm{L}^1(\mathbb{P}_{i,\rho})$ to*

$$\mathbb{E}_{j,\frac{\rho_{\mathrm{inv}}(j)}{\mathrm{tr}\,\rho_{\mathrm{inv}}(j)}}(T_j) = \big(\mathrm{tr}\ \rho_{\mathrm{inv}}(j)\big)^{-1}. \tag{19}$$

Let us discuss the proof of this theorem. First, (8) shows that if $\frac{1}{N_{j,n}} \sum_{k=1}^{N_{j,n}} \rho_{t_j^{(k)}} \underset{n\to\infty}{\to} \frac{\rho_{\mathrm{inv}}(j)}{\mathrm{tr}\,\big(\rho_{\mathrm{inv}}(j)\big)}$, which shows that we are in situation 1 of Theorem 3.23 and that consequently the operator $\mathfrak{P}_{j,j}$ is completely positive, preserves the trace by (17) and admits the faithful operator $\frac{\rho_{\mathrm{inv}}(j)}{\mathrm{tr}\,\big(\rho_{\mathrm{inv}}(j)\big)}$ as an invariant, so that it is irreducible. But if one denotes by $\mathcal{P}^{V\setminus\{j\}}(j,j)$ the set of loops from j to j that do not go through j except at their start- or end-points, the operator $\mathfrak{P}_{j,j}$ can be written $\mathfrak{P}_{j,j}(\varrho) = \sum_{\pi\in\mathcal{P}^{V\setminus\{j\}}(j,j)} L_\pi \varrho L_\pi^*$. The family $\phi_\pi : \varrho \mapsto L_\pi \varrho L_\pi^*$ for $\pi \in \mathcal{P}^{V\setminus\{j\}}(j,j)$ defines an instrument and the associated probability distribution (6) on $\big(\mathcal{P}^{V\setminus\{j\}}(j,j)\big)^{\mathbb{N}}$ is therefore ergodic by remark 3.1. We see that $t_j^{(k)}$ identifies with the total length of the first k elements of $\omega \in \big(\mathcal{P}^{V\setminus\{j\}}(j,j)\big)^{\mathbb{N}}$; it is therefore an additive functional, and one obtains the final result thanks to Birkhoff's ergodic Theorem.

The last topic considered by [P17] is the Dirichlet problem. This question being less natural in the context of repeated measurements than the previous ones, we will not discuss it here: let us simply say that the preceding results allow to construct solutions to problems of the type $\big((\mathrm{Id}-\Phi)(Z)\big)_i = A_i$ for i in a discrete domain D, and $Z_j = B_j$ for j on the boundary ∂D of D, where $(A_i)_{i\in D}$ and $(B_j)_{j\in\partial D}$ are the givens of the problem.

Remark 3.25. Once again, Propositions 3.16 and 3.17, Theorems 3.21, 3.23 and 3.24 s'extend immediately to the case of extended OQW. Besides, the decompositions of Φ into irreducible OQW in [P15] allow to obtain expressions for the quantities like $\mathbb{P}_{i,\varrho}(t_j < \infty)$ introduced above, without irreducibility assumptions. We will not detail these expressions here.

2. Entropy of repeated measurement statistics

This section describes the results obtained in the articles [P20][2] and [P26]. The goal of the article [P20] was to understand the entropy production of indirect measurements, or more loosely to describe "the appearance of the arrow of time": if one receives a list $(i_1,\ldots,i_n)$ which we know is a list of measurement outcomes, read either forward or backward, can one determine the correct direction?

We consider an instrument $\mathcal{I} = (\Phi_i, i \in V)$ on a Hilbert space $\mathcal{H}$ of finite dimension, and the probability measure $\mathbb{P}_\rho$ on $\Omega = V^{\mathbb{N}}$ defined by (1). To construct the reversed measure, we need to make the following assumptions:

(A) The initial state ρ is Φ-invariant and faithful: $\Phi(\rho) = \rho$ and ρ is definite-positive.

We consider an involution θ of V. Assumption **(A)** ensures that

$$\widehat{\mathbb{P}}_\rho(i_1,\ldots,i_n) = \mathbb{P}_\rho\big(\theta(i_n),\ldots,\theta(i_1)\big) \tag{20}$$

defines via Kolmogorov's extension Theorem a probability on $V^{\mathbb{N}}$ which is invariant by the left shift, which we denote by τ. Determining the arrow of time, as we discussed it above, amounts to define a statistical test discriminating between $\mathbb{P}_\rho$ and $\widehat{\mathbb{P}}_\rho$.

Remark 3.26. There exists at least one instrument $\widehat{\mathcal{I}}$ such that the measurement statistics associated with $(\widehat{\mathcal{I}},\widehat{\rho})$ is $\widehat{\mathbb{P}}_\rho$ with $\widehat{\rho} = \rho$. This instrument was first identified by Crooks, see [36], and is defined by $\widehat{\mathcal{I}} = (\widehat{\Phi}_i, i \in I)$ where

$$\widehat{\Phi}_i(\eta) = \rho^{+1/2}\Phi^*_{\theta(i)}(\rho^{-1/2}\eta\rho^{-1/2})\rho^{+1/2}. \tag{21}$$

We also have that $\widehat{\Phi}(\widehat{\rho}) = \widehat{\rho}$ if $\widehat{\Phi} := \sum_{i\in I}\widehat{\Phi}_i$.

To simplify the notation, we denote by $\mathbb{P}$ and $\widehat{\mathbb{P}}$ these two measures, and by $\mathbb{P}_n$, $\widehat{\mathbb{P}}_n$ respectively their marginals on V^n. To be able to define the random variable central to this section, we make an additional assumption:

[2] This article was written "in the Heisenberg interpretation", or in other words the ϕ_i of [P20] are our ϕ_i^*

(B) For all n one has $\operatorname{supp}\mathbb{P}_n = \operatorname{supp}\widehat{\mathbb{P}}_n$.

(remark that $\operatorname{supp}\mathbb{P}_n \neq \operatorname{supp}\widehat{\mathbb{P}}_n$ implies $\operatorname{supp}\mathbb{P}_{n+1} \neq \operatorname{supp}\widehat{\mathbb{P}}_{n+1}$). We then define unambiguously a random variable ς_n, equal $\mathbb{P}$-almost surely on Ω to

$$\varsigma_n(\omega) = \log \frac{\mathbb{P}(i_1, \dots, i_n)}{\widehat{\mathbb{P}}(i_1, \dots, i_n)}. \tag{22}$$

This variable is called the relative information random variable (see [35]), or the log-likelihood. It has the property that

$$\mathbb{E}(\varsigma_n) = S(\mathbb{P}_n | \widehat{\mathbb{P}}_n) \qquad \log \mathbb{E}(\mathrm{e}^{-\alpha\varsigma_n}) = S_{1-\alpha}(\mathbb{P}_n | \widehat{\mathbb{P}}_n) = S_\alpha(\mathbb{P}_n | \widehat{\mathbb{P}}_n) \tag{23}$$

(here and below, expectations denoted by $\mathbb{E}$ are computed with respect to $\mathbb{P}$). One can show the following results:

Theorem 3.27 ([P20]). *Assume that assumptions* **(A)** *and* **(B)** *hold. Then:*

1. *The limit*
$$\mathrm{ep} = \lim_{n\to\infty} \frac{1}{n}\,\mathbb{E}(\varsigma_n) \tag{24}$$
exists in $[0, +\infty]$. *This quantity is called the mean rate of entropy production.*

2. *The limit*
$$\sigma = \lim_{n\to\infty} \frac{1}{n}\,\varsigma_n \tag{25}$$
exists $\mathbb{P}$*-almost surely in* $\mathbb{R}$ *and satisfies* $\sigma \circ \tau = \sigma$. *If in addition* $\mathrm{ep} < \infty$ *then the limit* (25) *holds in the* $\mathrm{L}^1(\mathbb{P})$ *sense. The quantity* $\sigma(\omega)$ *is called the entropy production along the trajectory* $\omega \in \Omega$.

The first point of this proposition follows easily from Fekete's lemma on sub-additive sequences of reals. The second follows from Kingman's sub-additive ergodic Theorem (which can be seen as an almost sure version of Fekete's lemma, see Theorem 10.1 of [121]), and of the Shannon–McMillan–Breiman Theorem (Theorem 16.8.1 of [35]). The sub-additivity properties follows from the simple inequality:

$$\begin{aligned}
\mathbb{P}(i_1, \dots, i_{m+n}) &= \operatorname{tr}\left(\Phi_{i_m} \circ \dots \circ \Phi_{i_1}(\rho)\, \Phi^*_{i_{m+1}} \circ \dots \circ \Phi^*_{i_{m+n}}(\mathrm{Id})\right) \\
&\leq \operatorname{tr}\left(\Phi_{i_m} \circ \dots \circ \Phi_{i_1}(\rho)\right)\, \operatorname{tr}\left(\Phi^*_{i_{m+1}} \circ \dots \circ \Phi^*_{i_{m+n}}(\mathrm{Id})\right) \\
&\leq \lambda_0^{-1}\, \mathbb{P}(i_1, \dots, i_m)\mathbb{P}(i_{m+1}, \dots, i_{m+n})
\end{aligned} \tag{26}$$

where $\lambda_0 = \min \operatorname{sp} \rho$. One can then show that under an irreducibility assumption, the vanishing of ep characterises the equality of the measure $\mathbb{P}$ and of the reversed measure $\widehat{\mathbb{P}}$. We recall (see remark 3.1) that the dynamical system $(\Omega, \mathbb{P}, \tau)$ is ergodic if Φ if irreducible.

Theorem 3.28 ([P20]). *Suppose that the dynamical system* $(\Omega, \mathbb{P}, \tau)$ *is ergodic. Then* $\sigma = \mathrm{ep}$, *and one has* $\mathbb{P} = \widehat{\mathbb{P}}$ *if and only if* $\sigma = 0$ $\mathbb{P}$*-almost surely.*

When the instrument $\mathcal{I}$ is associated with a Markov chain of transition matrix $\Pi = (\Pi_{i,j})_{i,j\in V}$ as in remark 3.4, that is, $\mathcal{I} = (\Phi_{i,j}, i \in V)$ with $\mathcal{H} = \mathbb{C}$ and $\phi_{i,j} = \Pi_{j,i}$, and that ρ is the diagonal state of coefficients $(\rho_i, i \in V)$ which is supposed invariant by Π, thenn Birkhoff's ergodic Theorem shows immediately that

$$\sigma = \sum_{i,j\in V} \rho_i \Pi_{i,j} \log \frac{\Pi_{i,j}}{\Pi_{j,i}} = \sum_{i,j\in V} \rho_i \Pi_{i,j} \log \frac{\rho_i \Pi_{i,j}}{\rho_j \Pi_{j,i}},$$

so that $\sigma = 0$ if and only if the chain satisfies the detailed balance relation $\rho_i \Pi_{i,j} = \rho_j \Pi_{j,i}$. It is then natural to ask if there exists a similar characterisation in the general case. This question is the topic of the article [P26]. There exist various notions of quantum detailed balance in the literature, which concern the quantum channel Φ and not the instrument $(\Phi_i, i \in V)$. We choose to say that a quantum channel Φ satisfies the *quantum detailed balance* if, denoting $\tilde{\Phi}$ the dual of Φ for the scalar product $(\eta_1, \eta_2) \mapsto \operatorname{tr}(\rho^{\frac{1}{2}} \eta_1^* \rho^{\frac{1}{2}} \eta_2)$, there exists a unitary or antiunitary involution J such that $J\rho = \rho J$ and $J\Phi(\eta)J = \tilde{\Phi}(J\eta J)$ for all η (one can show that the other common notions of quantum detailed balance are equivalent to one another, and are stronger than this one). It is natural to extend this notion to instruments, saying that $(\Phi_i, i \in V)$ satisfies quantum detailed balance[3] if there exists such a J and an involution θ of V such that $\tilde{\Phi}_i = \Phi_{\theta(i)}$.

Stating a satisfactory equivalence requires the introduction of a new notion: an instrument $(\Phi_i, i \in V)$ is called *complete* if, when each Φ_i is written in the form $\Phi_i(\rho) = \operatorname{tr}_{\mathcal{H}_\mathcal{E}} \big(U(\rho \otimes \xi)U^* \Pi_i\big)$ with Π_i a nonnegative operator such that $\sum_{i\in V} \Pi_i = \mathrm{Id}_{\mathcal{H}_\mathcal{E}}$ (this is always possible, see Theorem 2.4 of [125]) and satisfying the additional condition $\operatorname{vect}(\Pi_i, i \in V) = \mathcal{B}(\mathcal{H}_\mathcal{E})$. In the case of indirect measurements as described in section 0.3, this completude assumption is not satisfied unless $\dim \mathcal{H}_\mathcal{E} = 1$. We would then need to extend the measurement protocol, assuming that there exists not just one measured observable M, but a family $M_j, j \in V$ of observables, which we assume once again for simplicity are nondegenerate, with spectral projectors $\pi_{i,j}, i = 1, \dots, \dim \mathcal{H}_\mathcal{E}$ and that at each iteration of the evolution-measurement cycle one chooses randomly which observable M_j will be measured, according to a probability distribution $(p_j)_{j\in J}$, so that the relevant instrument is given by the following $\Phi_{i,j}$:

$$\Phi_{i,j}(\rho) := p_j \operatorname{tr}_{\mathcal{H}_\mathcal{E}} \big((\mathrm{Id}_{\mathcal{H}_\mathcal{S}} \otimes \pi_{i,j})\, U(\rho \otimes \xi)U^* \,(\mathrm{Id}_{\mathcal{H}_\mathcal{S}} \otimes \pi_{i,j})\big).$$

The completude assumption then amounts to say that $\operatorname{vect}(\pi_{i,j}, i = 1, \dots, \dim \mathcal{H}_\mathcal{E}, j \in J) = \mathcal{B}(\mathcal{H}_\mathcal{E})$. Remark that this property is trivially satisfied by minimal dilations of Markov chains, for which $\dim \mathcal{H}_\mathcal{E} = 1$, see Remark 3.4.

This notion now allows to give the following equivalence:

THEOREM 3.29 ([P26]). *Suppose that the quantum channel Φ is irreducible. Then the following points are equivalent:*

1. *the channel Φ satisfies the quantum detailed balance for J;*
2. *there exist a complete instrument $(\Phi_i, i \in V)$ with $\Phi = \sum_{i\in V} \Phi_i$, and an involution θ of V, that satisfies quantum detailed balance for J and θ;*
3. *there exists a complete instrument $(\Phi_i, i \in V)$ with $\Phi = \sum_{i\in V} \Phi_i$, and an involution θ of V, such that the entropy production* ep *defined in Proposition 3.27 is null.*

The completude assumption allows to associate to the probability $\mathbb{P}_\rho$ induced by the instruments, as in (1), a unique state on the chain of environments $\mathcal{E}_1, \mathcal{E}_2, \dots$ and this state is a *finitely correlated state* in the sense of Fannes, Nachtergaele and Werner (see [59, 60]). One then benefits from uniqueness results

[3] One would naturally like to call this property the detailed quantum detailed balance.

(once again see [59, 60], as well as [74]). Theorem 3.29 shows that the detailed quantum balance for Φ is equivalent to the existence of an indirect measurement protocol which is complete in the sense of the information it provides, induces an average evolution equal to Φ, and has vanishing entropy production ep.

Let us now return to the question of discriminating between $\mathbb{P}$ and $\widehat{\mathbb{P}}$; Theorem 3.27 allows us to give a first answer. In Theorems 3.30 and 3.33, we build for each n, a "test", an event T_n and we will apply the decision rule that observing T_n leads us to conclude that the data was sampled from $\mathbb{P}$, and not observing T_n to conclude that they were sampled from $\widehat{\mathbb{P}}$. Under this decision rule, $\mathbb{P}_n(T_n^c)$ is the error "to conclude erroneously to $\widehat{\mathbb{P}}$" and $\widehat{\mathbb{P}}_n(T_n)$ the error "to conclude erroneously to $\mathbb{P}$".

Theorem 3.30 ([P20]). *Suppose that assumptions* **(A)** *and* **(B)** *hold, and let* $\epsilon \in (0,1)$. *For* $n \in \mathbb{N}^*$ *we define*

$$s_n(\epsilon) = \min\big(\widehat{\mathbb{P}}_n(T_n)\ \textit{pour}\ T_n \in \Omega_n\ \textit{with}\ \mathbb{P}_n(T^c) \leq \epsilon\big).$$

If the dynamical system $(\Omega, \mathbb{P}, \tau)$ *is ergodic, then*

$$\lim_{n\to\infty} \frac{1}{n} \log s_n(\epsilon) = -\mathrm{ep}.$$

This result shows that the rate of decrease of $\widehat{\mathbb{P}}_n(T_n)$ optimal under the constraint that $\mathbb{P}_n(T_n^c)$ remains smaller than ϵ is ep (and one can replace, in this sentence, "remains smaller than ϵ" by "tends to zero"). This is a standard application of Theorem 3.27, see section 3.4 of [45].

To be more precise in terms of determining the arrow of time, we need to introduce new hypotheses. The assumption given below is not exactly the one given in [P20] but one shows easily that it is equivalent to it (see [18]).

(C) There exist $C > 0$ and $\tau \geq 0$ such that for all $i_1, \ldots, i_m$ and $j_1, \ldots, j_n$ there exist $k_1, \ldots, k_\ell$ with $\ell \leq \tau$, such that

$$\mathbb{P}(i_1, \ldots, i_m, k_1, \ldots, k_\ell, j_1, \ldots, j_n) \geq C\, \mathbb{P}(i_1, \ldots, i_m)\, \mathbb{P}(j_1, \ldots, j_n)$$
$$\widehat{\mathbb{P}}(i_1, \ldots, i_m, k_1, \ldots, k_\ell, j_1, \ldots, j_n) \geq C\, \widehat{\mathbb{P}}(i_1, \ldots, i_m)\, \widehat{\mathbb{P}}(j_1, \ldots, j_n).$$

Remark 3.31. One can show (see Proposition 2.8 of [62]) that if Φ is irreducible, then the above condition is true for $\mathbb{P}$ alone (and the same is true of $\widehat{\mathbb{P}}$). This is enough for example to study the Rényi entropies of $\mathbb{P}$ and $\widehat{\mathbb{P}}$, but here we are interested in the relative properties relatives of $\mathbb{P}$ and $\widehat{\mathbb{P}}$ and will need condition **(C)** that allows to make a common choice of $k_1, \ldots, k_\ell$ for $\mathbb{P}$ and for $\widehat{\mathbb{P}}$.

We will also consider a strengthening of **(C)** that consists in imposing $\ell = 0$:

(D) There exists $C > 0$ such that for all $i_1, \ldots, i_m$ and $j_1, \ldots, j_n$,

$$\mathbb{P}(i_1, \ldots, i_m, j_1, \ldots, j_n) \geq C\, \mathbb{P}(i_1, \ldots, i_m)\, \mathbb{P}(j_1, \ldots, j_n)$$
$$\widehat{\mathbb{P}}(i_1, \ldots, i_m, j_1, \ldots, j_n) \geq C\, \widehat{\mathbb{P}}(i_1, \ldots, i_m)\, \widehat{\mathbb{P}}(j_1, \ldots, j_n).$$

One can then also prove:

Theorem 3.32 ([P20]). *Suppose that assumptions* **(A)**, **(B)** *and* **(C)** *hold. Then:*

1. *For all* $\alpha \in (0,1)$, *the limit*

$$e(\alpha) = \lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}(\mathrm{e}^{-\alpha\varsigma_n}) \tag{27}$$

exists in $(-\infty, +\infty]$. *The function* e *is convex, vanishes at* 0 *and* 1, *and satisfies the symmetry*

$$e(\alpha) = e(1-\alpha), \qquad \alpha \in \mathbb{R}. \tag{28}$$

It is differentiable on $(0,1)$ *and admits a right-derivative at* 0 *and a left-derivative at* 1, *that are respectively equal to* $-\mathrm{ep}$ *and* $+\mathrm{ep}$. *The sequence* $(\varsigma_n)_n$ *satisfies a local large deviations principle on* $(-\mathrm{ep}, +\mathrm{ep})$.

2. *If in addition* **(D)** *holds, then* e *is finite and differentiable on* $\mathbb{R}$. *The sequence* $(\varsigma_n)_n$ *satisfies a (global) large deviations principle for the rate function*

$$I(s) = \sup_{\alpha\in\mathbb{R}} (\alpha s - e(-\alpha)), \qquad s \in \mathbb{R},$$

that satisfies the relation $I(-s) = I(s) + s$ *for all* $s \in \mathbb{R}$.

These results were later strengthened by [38], where the authors show that assumptions **(A)**, **(B)** and **(C)** suffice for the existence in $(-\infty, +\infty]$ of the quantity $e(\alpha)$ defined by (27), for all $\alpha \in \mathbb{R}$, and for a large deviations principle to hold. Their approach of large deviations uses the Ruelle–Lanford theory and not the Gärtner–Ellis Theorem, which we used in [P20].

In terms of determination of the arrow of time, Theorem 3.32 has the following consequence:

THEOREM 3.33 ([P20]). *Suppose that assumptions* **(A)**, **(B)** *and* **(C)** *hold. Then:*

1. *If one defines the total error*

$$c_n = \min_{T_n \in \Omega_n} \big(\widehat{\mathbb{P}}_n(T_n) + \mathbb{P}_n(T_n^c)\big),$$

in the test of $\mathbb{P}$ *versus* $\widehat{\mathbb{P}}$, *then one has*

$$\lim_{n\to\infty} c_n = \min_{\alpha\in[0,1]} e(\alpha) = e\big(1/2\big). \tag{29}$$

2. *If one defines for* $s \geq 0$

$$h(s) = \inf\big\{ \lim_{n\to\infty} \tfrac{1}{n} \log \widehat{\mathbb{P}}_n(T_n),\ T_n \in \Omega_n \textit{ such that } \limsup_{n\to\infty} \tfrac{1}{n} \log \widehat{\mathbb{P}}_n(T_n^c) < -s \big\},$$

the decrease rate $\widehat{\mathbb{P}}_n(T_n)$ *optimal under the constraint that* $\widehat{\mathbb{P}}_n(T_n^c)$ *decreases with a rate* s, *then*

$$h(s) = \inf_{\alpha\in[0,1)} \frac{s\alpha + e(\alpha)}{1-\alpha} \tag{30}$$

(and the latter quantity vanishes if $s > \mathrm{ep}$*).*

Remark 3.34. The goal of article [P20] was the study of the appearance of the arrow of time in indirect measurements, and this is why, when we mention the instrument $\widehat{\mathcal{I}}$ and the state $\widehat{\rho}$, we alway assume that they induce the time-reversal of $\mathbb{P}_\rho$. One can however almost immediately extend our results to discuss hypothesis testing between probabilities $\mathbb{P}_\rho$ and $\widehat{\mathbb{P}}_{\widehat{\rho}}$ induced by $(\mathcal{I}, \rho)$ and $(\widehat{\mathcal{I}}, \widehat{\rho})$ respectively, in a more general situation. Indeed, Theorems 3.27 and 3.30, and as a consequence Theorems 3.32 and 3.33, remain true if one replaces **(A)** by the assumption that $\Phi(\rho) = \rho$ and $\widehat{\Phi}(\widehat{\rho}) = \widehat{\rho}$ with ρ and $\widehat{\rho}$ faitfhul, with the exception that (28) and the last identity "$= e(1/2)$" of (29) no longer hold in general.

We must say a word of the proofs of Theorem 3.32 (Theorem 3.33 is then a standard application, see section 3.5 of [45]). This proof uses the framework of non-additive thermodynamic formalism. This consists in showing that $e(\alpha)$ can be written as the supremum of a certain functional on the set of shift-invariant probability measures on $V^{\mathbb{N}}$; this is shown by exploiting a variational principle, then showing that the maximum is attained, which can be done constructing explicit optima using the Kolmogorov–Sinai Theorem. One can then show, following the proof of Theorem 1.2 of [61] that the left- and right-derivatives of $e(\alpha)$ can be written as the infimum and supremum of another function on the set of shift-invariant probability measures on which the previous supremum is attained. All of this uses only the "upper decoupling" proven in (26). We then show that this latter set contains a unique element using the "lower decoupling" property implied by assumption **(C)**.

Last, we remark that the above theorems rely on assumptions (**(B)** and most of all **(C)** or **(D)**) which we still have not proved for non-trivial models. The article [P20] did not discuss any model, all examples being postponed to the article [18] (to which I did not contribute). The latter article proposes many models serving as examples for the results of [P20] and [38], and as counter-examples to possible stronger statements. Here we will mention two examples: classical Markov chains, and two-time measurements.

Example 3.35. We consider a Markov chain on $V = \{1, \dots, \ell\}$ with transition matrix $\Pi = (\Pi_{i,j})$ and invariant probability π. One can construct (as in remark 3.4) an instrument such that the probability $\mathbb{P}_\rho$ associated with (1) is the distribution of this Markov chain, by letting $\Phi_i(\rho) = \sum_j \Pi_{j,i}\rho_{j,j}|i\rangle\langle i|$ for all $i \in V$. Then, if the chain is irreducible, **(A)** holds, and if one has $\Pi_{i,j} = 0 \Leftrightarrow \Pi_{\theta(j),\theta(i)} = 0$ then **(B)** and **(C)** hold. Remark in addition that in this case one can actually prove directly that $e(\alpha)$ is well-defined and smooth on $\mathbb{R}$ by classical techniques similar to what was done on page 62.

Example 3.36. We consider a setting of two-time indirect measurement of an observable M: this amounts to add to the protocol described in section 0.3 another step preceding step 1, which is to measure M before the interaction with $\mathcal{H}_\mathcal{S}$ (which is denoted $\mathcal{H}$ here). The outcome of this new protocol is then a pair (i, j) of elements of $\operatorname{sp} M$, and the corresponding instrument is $(\Phi_{i,j},\ i, j \in \operatorname{sp} M)$, with

$$\Phi_{i,j}(\rho) = \operatorname{tr}_\mathcal{K}\big(U(\rho \otimes \Pi_i\xi)U^*(\mathrm{Id} \otimes \Pi_j)\big).$$

Then [17] shows that if ξ is definite-positive then there exists a definite-positive ρ with $\Phi(\rho) = \rho$; if one considers such a ρ as an initial state, then assumption **(A)** holds. We will say that the considered system is time-reversal invariant (TRI) if there exist two unitaries $W_\mathcal{H}$ and $W_\mathcal{K}$ of $\mathcal{H}$ and $\mathcal{K}$ respectively, such that $[M, W_\mathcal{K}] = 0$ and $(W_\mathcal{H} \otimes W_\mathcal{K})U(W_\mathcal{H} \otimes W_\mathcal{K}) = U^*$. One can then show (see [18]) that if the system is TRI, and if one chooses the involution $\theta(i, j) = (j, i)$, then assumption **(B)** holds, and if in addition Φ is irreducible, then assumption **(C)** also holds. One can then prove that $e(\alpha)$ is well-defined and smooth on $\mathbb{R}$, once again using the same techniques as on page 62, and that $e(\alpha)$ is equal to the logarithm of the spectral radius of

$$\Phi^{(\alpha)} = \sum_{i,i' \in \operatorname{sp} M} \Big(\frac{\operatorname{tr}(\Pi_i\xi)}{\operatorname{tr}(\Pi_{i'}\xi)}\Big)^\alpha \Phi_{i,i'}. \tag{31}$$

We will return to this example in chapter 4, where we consider repeated measurements as in the present example, but with time-dependent parameters.

3. Repeated measurements of a system: the system

We now turn to the study of the behaviour of the random state ρ_n after n indirect measurements, when the number n of measurements tends to infinity, with the aim to prove a convergence result. Remark immediately that the known results on Markov chains (which are a special case of the present situation, see remark 3.4) indicate that one can at most hope for a convergence in distribution. Besides, when Theorem 3.3 applies, a convergence in distribution of $(\rho_n)_n$ can only be to a pure state. We will therefore be interested in the process $(x_n)_n$ induced on pure states (in the sense of relation (4)) which we will view as points on the projective sphere. Let us introduce some notation: for $x \in \mathcal{H}$, we denote by $\hat{x}$ the corresponding class in the projective space $\mathrm{P}\mathcal{H}$. For L an operator on $\mathcal{H}$, we denote if $Lx \neq 0$ by $L \cdot \hat{x}$ the class $\widehat{Lx}$. We will generalize the setting of our study by considering a measure μ on $\mathcal{B}(\mathcal{H})$ (μ is in general not finite), equipped with its Borel σ-algebra, and we assume that μ satisfies

$$\int L^* L \, \mathrm{d}\mu(L) = \mathrm{Id}_{\mathcal{H}}. \tag{32}$$

We then define

$$\Phi(\rho) = \int L \rho L^* \, \mathrm{d}\mu(L). \tag{33}$$

The map Φ is completely positive and (32) implies that it preserves the trace. The case considered previously of perfect instruments $(\Phi_i)_{i \in V}$ with V finite, where $\Phi_i(\eta) = L_i \eta L_i^*$ for all $i \in V$ corresponds to the case where μ has finite support, and where μ is the image by $i \mapsto L_i$ ot the distribution on the measurement outcomes i (in which case (32) is the standard condition $\sum L_i^* L_i = \mathrm{Id}_{\mathcal{H}}$); the reader can keep this example in mind.

Similar to (2) one could define a stochastic process on $\Omega = \mathcal{B}(\mathcal{H})^{\mathbb{N}}$ (with a probability measure defined below) by

$$\rho_n(\omega) = \frac{L_n \dots L_1 \rho L_1^* \dots L_n^*}{\operatorname{tr}\left(L_n \dots L_1 \rho L_1^* \dots L_n^*\right)}$$

if $\omega = (L_1, L_2, \dots)$, and this process would have the property that ρ_n is pure for all n is ρ is pure. Because we will consider a situation where this process $(\rho_n)_n$ would be almost-surely asymptotically pure, we directly work with pure states and view them as elements of the projective space $\mathrm{P}\mathcal{H}$.

Define an assumption analogous to that of Theorem 3.3 (we recall that $X \propto Y$ means that X and Y are proportional to one another, i.e. $X = cY$ or $cX = Y$):

(Pur) Any orthogonal projector π such that for all $n \in \mathbb{N}$, $\pi L_1^* \dots L_n^* L_n \dots L_1 \pi \propto \pi$ for $\mu^{\otimes n}$-almost all $(L_1, \dots, L_n)$, is of rank 0 or 1.

Remark 3.37. When $\dim \mathcal{H} = 2$, assumption **(Pur)** is equivalent to saying that μ does not have support in the set of unitary operators. In higher dimension, this essential assumption is difficult to check, even when μ has finite support. Remark **??** will show that this is hardly suprising.

We will now consider the Markov chain $(\hat{x}_n)_n$ on $\mathrm{P}\mathcal{H}$ associated with the transition matrix

$$\Pi(\hat{x}, S) = \int_{\mathcal{B}(\mathcal{H})} \mathbf{1}_S(L \cdot \hat{x}) \|Lx\|^2 \, \mathrm{d}\mu(L) \tag{34}$$

(where x is an arbitrary representative of $\hat{x}$). A natural first step in the analysis of the asymptotic behaviour of the distribution of $\hat{x}_n$ is to study the invariant probability measure(s) of Π. Remark that, $\mathrm{P}\mathcal{H}$ being compact, and the kernel Π being Feller, the Markov–Kakutani theorem shows that there exists at least one invariant probability measure. We will show that if assumption **(Pur)** holds, and that Φ is irreducible, then this invariant probability is unique. We will then show the convergence of $(\hat{x}_n)_n$ to that invariant probability, which will be exponentially fast for the first Wasserstein distance.

Remark 3.38. Our Markov chain can be written in the form $L_n \dots L_1 \cdot \hat{x}_0$, where the operators L_i are random. The present study therefore fits in the general framework of random products of matrices. Such products have been widely studied in the case where the L_i are invertible, independent and identiquely distributed, see [26, 63, 72, 91]. The article [71] however studies the case where the choice of L_{n+1} conditional on x_n is done with a probability proportional to $\|L_{n+1}x_n\|^s$. Our transition (34) therefore corresponds to the case $s = 2$. We compare our assumptions to those of [71] in remark **??** below.

To state our main result, we define a metric on $\mathrm{P}\mathcal{H}$ by

$$\operatorname{dist}(\hat{x}, \hat{y}) = \left(1 - \left|\tfrac{\langle x,y\rangle}{\|x\|\,\|y\|}\right|^2\right)^{\frac{1}{2}}, \tag{35}$$

where x and y are representatives of norm 1 of $\hat{x}$ and $\hat{y}$. The Wasserstein distance of order 1 between two probability measures on $\mathrm{P}\mathcal{H}$ can then be defined using the Kantorovich–Rubinstein duality by

$$W_1(\sigma, \tau) = \sup_{f \in \mathrm{Lip}_1(\mathrm{P}\mathcal{H})} \Big| \int_X f \, \mathrm{d}\sigma - \int_X f \, \mathrm{d}\tau \Big|,$$

where $\mathrm{Lip}_1(\mathrm{P}\mathcal{H}) = \{f : \mathrm{P}\mathcal{H} \to \mathbb{R} \text{ such that } |f(x) - f(y)| \leq \operatorname{dist}(x, y)\}$ is the set of Lipschitz functions with constant at most 1. The main result of [P24] is the following:

THEOREM 3.39 ([P24]). *Assume that μ satisfies the assumption* **(Pur)** *and that Φ admits a unique invariant state. Then Π admits a unique invariant probability measure ν_{inv} and there exist $m \in \{1, \dots, \dim \mathcal{H}\}$, $C > 0$ and $0 < \lambda < 1$ such that for any probability measure ν on $(\mathrm{P}\mathcal{H}, \mathcal{B})$,*

$$W_1\Big(\frac{1}{m} \sum_{r=0}^{m-1} \nu \Pi^{mn+r}, \nu_{\mathrm{inv}}\Big) \leq C\lambda^n. \tag{36}$$

Remark 3.40. The convergence in the distance W_1 is equivalent to convergence in distribution by the compactness of $\mathrm{P}\mathcal{H}$. We can however not obtain a convergence in total variation: suppose for example that μ has finite support in the set of invertible operators. Then if ν_a is an atomic measure and ν_b a diffuse measure, the total variation $\|\nu_a\Pi^n - \nu_b\Pi^n\|_{\mathrm{VT}}$ equals 1 for all n; one can therefore not have $\|\nu_a\Pi^n - \nu_{\mathrm{inv}}\|_{\mathrm{VT}} \underset{n\to\infty}{\to} 0$ and $\|\nu_b\Pi^n - \nu_{\mathrm{inv}}\|_{\mathrm{VT}} \underset{n\to\infty}{\to} 0$.

Remark 3.41. If one assumes only **(Pur)** and not the uniqueness of the invariant state, the set of invariant probability measures of Π is a simplex, whose extremal points are the invariant probabilities associated with irreducible components of Φ. This is shown in the annex B of [P24], and uses the decompositions of channels developed in [P14]. One can similarly show a convergence of the type (36) to an invariant measure ν_{inv}, but that depends on the initial distribution of $\hat{x}_0$.

Remark 3.42. We know absolutely nothing about the properties of the invariant probability measure ν_{inv}.

To prove Theorem 3.39, we consider jointly the pure state, element of $\mathrm{P}\mathcal{H}$, and the "measurement outcomes" represented here by $\Omega = \mathcal{B}(\mathcal{H})^{\otimes\mathbb{N}}$. We equip $\mathrm{P}\mathcal{H}$ with its Borel σ-algebra $\mathcal{B}$ and Ω with the cylindrical σ-algebra $\mathcal{C}$ (we denote by $\mathcal{C}_n$ the σ-algebra of events that depend only on the first n coordinates). An element ω of Ω can then be written $(v_1, v_2, \dots)$; we denote $L_1, L_2, \dots$ the coordinate mappings and $K_n := L_n \dots L_1$. We identify the $\mathcal{C}$-measurable functions (such as L_n or K_n) to $\mathcal{B} \otimes \mathcal{C}$-measurable functions that do not depend on the variable $\hat{x}$. For ν a probability measure on $\mathrm{P}\mathcal{H}$, we define a measure $\mathbb{Q}^\nu$ on $(\mathrm{P}\mathcal{H}, \mathcal{B})$ by

$$\mathbb{Q}^\nu(S \times O_n) := \int_{S \times O_n} \|K_n(\omega)x\|^2 \mathrm{d}\nu(\hat{x}) \mathrm{d}\mu^{\otimes n}(\omega) \tag{37}$$

for all $S \in \mathcal{B}$ and $C \in \mathcal{C}_n$. Once again, the property (32) ensures that (37) defines a probability measure on $\mathrm{P}\mathcal{H} \times \Omega$. In addition, the marginal of $\mathbb{Q}^\nu$ on $\mathcal{B}$ is by construction ν. We will express the marginal of $\mathbb{Q}^\nu$ on $\mathcal{C}$; for this let us define

$$\rho_\nu = \mathbb{E}^\nu\left(|\hat{x}\rangle\langle\hat{x}|\right)$$

where $|\hat{x}\rangle\langle\hat{x}|$ is $|x\rangle\langle x|$ for any representative x of $\hat{x}$, and $\mathbb{E}^\nu$ is the expectation with respect to $\mathbb{Q}^\nu$. Define for $\rho \in \mathcal{S}(\mathcal{H})$ a probability measure on Ω (analogous to $\mathbb{P}_\rho$ of (3.1)) by letting for all $O_n \in \mathcal{C}_n$

$$\mathbb{P}_\rho(O_n) := \int_{O_n} \operatorname{tr}\left(K_n(\omega)\rho K_n^*(\omega)\right) \mathrm{d}\mu^{\otimes n}(\omega). \tag{38}$$

Then in particular, for all $S \in \mathcal{B}$ and $A \in \mathcal{C}$,

$$\mathbb{Q}^\nu(S \times A) = \int_S \mathbb{P}_{|\hat{x}\rangle\langle\hat{x}|}(A)\, \mathrm{d}\nu(\hat{x}). \tag{39}$$

Since one also sees easily that $\Phi(\rho_\nu) = \rho_{\nu\Pi}$, one has the following result:

Proposition 3.43 ([P24]). *$\mathbb{P}_{\rho_\nu}$ is the marginal of $\mathbb{Q}^\nu$ on $\mathcal{C}$, and if ν is Π-invariant then ρ_ν is an invariant state of Φ.*

In particular, if Φ admits a unique invariant state ρ_{inv}, then for all Π-invariant ν, the marginal on $\mathcal{C}$ of $\mathbb{Q}^\nu$ is $\mathbb{P}_{\rho_{\mathrm{inv}}}$.

A crucial element for the rest of the proof is the introduction of a well-chosen martingale. This martingale is given by

$$M_n := \frac{K_n^* K_n}{\operatorname{tr}(K_n^* K_n)} \tag{40}$$

which defines M_n in the $\mathbb{Q}^\nu$-almost sure sense, for all ν. Indeed, if one denotes by $\mathbb{P}_{\mathrm{ch}}$ the probability $\mathbb{P}_\rho$ for $\rho = \frac{1}{\dim \mathcal{H}} \mathrm{Id}$, one sees immediately that $\mathbb{P}_{\mathrm{ch}}\big(\operatorname{tr}(K_n^* K_n) = 0\big) = 0$. But, on the one hand, any $\rho \in \mathcal{S}(\mathcal{H})$ is dominated by $\|\rho\|\mathrm{Id}$ and therefore the probability $\mathbb{P}_\rho$ is absolutely continuous with respect to that $\mathbb{P}_{\mathrm{ch}}$; on the other hand, K_n being $\mathcal{C}_n$-measurable, one has $\mathbb{Q}^\nu\big(\operatorname{tr}(K_n^* K_n) = 0\big) = \mathbb{P}_{\rho_\nu}\big(\operatorname{tr}(K_n^* K_n) = 0\big) = 0$.

One can then show the following property for $(M_n)_n$:

Proposition 3.44 ([P24]). *For any probability ν on $(\mathrm{P}\mathcal{H}, \mathcal{B})$, the process (M_n) converges $\mathbb{Q}^\nu$-a.s. and L^1 to a $\mathcal{C}$-measurable random variable M_∞, and one has for all $\rho \in \mathcal{S}(\mathcal{H})$*

$$\frac{\mathrm{d}\mathbb{P}_\rho}{\mathrm{d}\mathbb{P}_{\mathrm{ch}}} = \dim \mathcal{H} \times \operatorname{tr}(\rho M_\infty). \tag{41}$$

In addition, the measure μ satisfies **(Pur)** *if and only if for all probability measures ν, the limit M_∞ is $\mathbb{Q}^\nu$-almost surely a projection of rank 1.*

The proof amounts to proving that $(M_n)_n$ is a $\mathbb{P}_{\mathrm{ch}}$-martingale; because it is a process of bounded operators on the finite-dimensional space $\mathcal{H}$, it converges $\mathbb{P}_{\mathrm{ch}}$-a.s. and as a consequence $\mathbb{Q}^\nu$-a.s. for all ν. The equivalence between **(Pur)** and the fact that M_∞ is $\mathbb{Q}^\nu$-a.s. of rank 1 is more involved technically and we refer the reader to [P24].

One can now construct a $(\mathcal{C}_n)_n$-adapted process which approximates $(\hat{x}_n)_n$. Indeed, K_n admits a polar decomposition $K_n = U_n D_n$ where $D_n = \left(\operatorname{tr}(K_n^* K_n)\right)^{1/2} M_n^{1/2}$. Since $\operatorname{tr}(K_n^* K_n)$ is $\mathbb{Q}^\nu$-a.s.

non null, one has $\hat{x}_n = (U_n M_n^{1/2}) \cdot \hat{x}_0$. If **(Pur)** is satisfied, then $M_n^{1/2}$ converges $\mathbb{Q}^\nu$-a.s. to a random projector $|\hat{z}\rangle\langle\hat{z}|$. Yet the relations (39) and (41) imply

$$\frac{\mathrm{d}\mathbb{Q}^\nu}{\mathrm{d}\nu \otimes \mathrm{d}\mathbb{P}_{\mathrm{ch}}} = \dim \mathcal{H} \times \left|\langle x_0, z\rangle\right|^2$$

and as a consequence $\mathbb{Q}^\nu(\langle x_0, z\rangle = 0) = 0$. We therefore have $\lim_{n\to\infty} \mathrm{dist}(\hat{x}_n, U_n \cdot \hat{z}) = 0$. This is enough to prove the uniqueness of the invariant measure: indeed, $U_n \cdot \hat{z}$ being $\mathcal{C}$-measurable, its distribution is uniquely determined by the fact that $\hat{x}_0$ follows a Π-invariant distribution. Yet at the same time $U_n \cdot \hat{z}$ approximates $\hat{x}_n$ which has the same distribution as $\hat{x}_0$. As a consequence there can exist only one Π-invariant distribution.

We know that there exists a unique invariant probability ν_{inv}. We now prove the exponential convergence (36). A first step uses the structure of the quantum channel Φ such as described in the annex A: that annex tells us that there exists $m \in \mathbb{N}^*$ (the period of Φ) such that the peripheral spectrum of Φ is the set of m-th roots of unity. This implies immediately that for all state η, one has

$$\Big\| \frac{1}{m} \sum_{r=0}^{m-1} \Phi^{mn+r}(\eta) - \rho_{\mathrm{inv}} \Big\| \leq C\lambda^n \tag{42}$$

where λ is the module of the largest non peripheral eigenvalue. Introduce now the left shift τ on Ω; one sees immediately that for all $\mathcal{C}$-measurable function f,

$$\mathbb{E}_\rho(f \circ \tau) = \mathbb{E}_{\Phi(\rho)}(f).$$

One can then deduce (that is Proposition 3.4 of [P24]) that for any probability ν on $\mathrm{P}\mathcal{H}$ and any $\mathcal{C}$-measurable essentially bounded function f,

$$\Big| \mathbb{E}^\nu\big(\frac{1}{m} \sum_{r=0}^{m-1} f \circ \tau^{mn+r}\big) - \mathbb{E}_{\rho_{\mathrm{inv}}}(f) \Big| \leq C \|f\|_\infty \lambda^n \tag{43}$$

for some C and λ that depend only on Φ.

A second step introduces the processes

$$\hat{z}_n(\omega) = \underset{\hat{x} \in \mathrm{P}\mathcal{H}}{\mathrm{argmax}}\ \|K_n x\|^2$$

and

$$\hat{y}_n = K_n \cdot \hat{z}_n.$$

Another way of defining $\hat{y}_n$ is as the equivalence class of the vector with maximal norm in the image of the unit sphere by K_n. We will show that $\hat{y}_n$ approximates $\hat{x}_n$ at an exponential rate, the intuition being that by definition the distribution of K_n favors the large values of $K_n x_0$. For this we will use the exterior product of vectors and operators, regarding which we only recall two useful properties: first, if A is an operator on $\mathcal{H}$ then the operator $\wedge^2 A$ on $\mathcal{H} \wedge \mathcal{H}$ defined by $(\wedge^2 A) : x \wedge y \mapsto Ax \wedge Ay$ has operator norm equal to the product of the largest two singular values of A, i.e. of the largest two eigenvalues (counted with multiplicity) of $(A^* A)^{1/2}$. Second, the distance (35) satisfies $\mathrm{dist}(\hat{x}, \hat{y}) = \frac{\|x \wedge y\|}{\|x\|\,\|y\|}$ (we refer the reader to chapter XVI of [96] for more details on the exterior products).

Exploiting the fact that $\|K_n z_n\| = \|K_n\|$, one can then show that

$$\mathbb{E}^\nu\big(\operatorname{dist}(\hat{x}_n, \hat{y}_n)\big) \le \int_{\mathrm{M}_k(\mathbb{C})^n} \|\wedge^2 (L_n \dots L_1)\| \,\mathrm{d}\mu^{\otimes n}(L_1, \dots, L_n).$$

Denote by $f(n)$ the upper bound; it can be written as $\dim \mathcal{H} \times \mathbb{E}_{\mathrm{ch}}\big(\frac{\wedge^2 K_n}{\operatorname{tr}(K_n^* K_n)}\big)$, and the integrand in this last expectation is proportional to the largest two eigenvalues of $M_n^{1/2}$, and therefore tends to zero $\mathbb{P}_{\mathrm{ch}}$-a.s. while remaining bounded, and therefore $f(n) \underset{n\to\infty}{\longrightarrow} 0$. Besides, one shows easily that $f(n)$ is sub-multiplicative; by Fekete's Lemma one obtains $f(n) \le \lambda^n$ for some $\lambda \in (0,1)$. We have therefore shown that $\mathbb{E}^\nu\big(\operatorname{dist}(\hat{x}_n, \hat{y}_n)\big) \le \lambda^n$; by the Markov property of $(\hat{x}_n)_n$ one then has for all $\ell \in \mathbb{N}$

$$\mathbb{E}^\nu\big(\operatorname{dist}(\hat{x}_{n+\ell}, \hat{y}_n \circ \tau^\ell)\big) \le \lambda^n. \tag{44}$$

One can now conclude the proof of Theorem 3.39: let therefore f be an element of $\mathrm{Lip}_1(\mathrm{P}\mathcal{H})$. We will approximate $\hat{x}_{mn+r}$ by $\hat{y}_{mp} \circ \theta^{mq+r}$ where $p = \lfloor \frac{n}{2} \rfloor$ and $q = \lceil \frac{n}{2} \rceil$, so that $p + q = n$. Then

$$\begin{aligned}
&\Big|\mathbb{E}^\nu\big(\frac{1}{m}\sum_{r=0}^{m-1} f(\hat{x}_{mn+r})\big) - \mathbb{E}_{\nu_{\mathrm{inv}}}\big(f(\hat{x}_0)\big)\Big| \\
&\qquad\le \frac{1}{m}\sum_{r=0}^{m-1} \big|\mathbb{E}^\nu\big(f(\hat{x}_{m(p+q)+r})\big) - \mathbb{E}^\nu\big(f(\hat{y}_{mp} \circ \theta^{mq+r})\big)\big| \\
&\qquad\quad + \frac{1}{m}\sum_{r=0}^{m-1} \big|\mathbb{E}_{\nu_{\mathrm{inv}}}\big(f(\hat{y}_{mp} \circ \theta^{mq+r})\big) - \mathbb{E}_{\nu_{\mathrm{inv}}}\big(f(\hat{x}_{m(p+q)+r})\big)\big| \\
&\qquad\quad + \Big|\frac{1}{m}\sum_{r=0}^{m-1} \mathbb{E}^\nu\big(f(\hat{y}_{mp} \circ \theta^{mq+r})\big) - \mathbb{E}_{\nu_{\mathrm{inv}}}\big(f(\hat{y}_{mp})\big)\Big|.
\end{aligned}$$

One can bound the first two terms in the upper bound using (44) and the fact that f is 1-Lipschitz. Since the quantity we are bounding is invariant by translation of f, one can assume that $\|f\|_\infty \le 1$; the last term is therefore upper bounded by (43). We obtain in the end

$$\Big|\mathbb{E}^\nu\Big(\frac{1}{m}\sum_{r=0}^{m-1} f(\hat{x}_{mn+r})\Big) - \mathbb{E}_{\nu_{\mathrm{inv}}}\big(f(\hat{x}_0)\big)\Big| \le 3C\lambda^{\lfloor \frac{n}{2} \rfloor}, \tag{45}$$

which proves Theorem 3.39.

Remark 3.45. The λ in inequality (45) is the largest of the two λ that appeared earlier, one in (43) and one in (44). The first is the largest non-peripheral eigenvalue of Φ, and its value is accessible. The second, on the other hand, was obtained from an application of Fekete's Lemma, and is therefore in general unknown. We therefore do not know how to explicit the λ in Theorem 3.39.

Remark 3.46. The results of [P24] allow us to give an extension of the ergodic Theorem of Kümmerer–Maassen (Theorem 3.2) to non-linear functionals of ρ_n; this will be done in a later publication.

4. Continous time open quantum walks

We now describe the results from [P25], which are interested in an analogue of Theorems 3.21 and 3.23 for continuous time open quantum walks. The definition of continous time OQW is due to Pellegrini in [107]. The space parameter is still constrained to a discrete set V (contrarily to the case of the "open Brownian motion" of Bauer, Bernard and Tilloy in [16]): the continous time OQW are to the OQW discussed in section 1 what continuous time Markov chains are to discrete time Markov chains (see however remark 3.47). These continuous time OQW are processes of continuous measurement, but are jump processes, and this greatly simplifies their construction with respect to the more general case considered in section 5 because one can use the counting processes introduced by Davies in [41].

We therefore consider again a discrete V, a Hilbert space of the form $\mathcal{H} = \bigoplus_{i\in V} \mathfrak{h}_i$, and operators $L_{i,j}$ from $\mathfrak{h}_j$ to $\mathfrak{h}_i$ for all i, j de V, as well as self-adjoint operators H_i on $\mathfrak{h}_i$ for all $i \in V$. We define for all $t \in \mathbb{R}_+$ the set

$$\Xi_t^{(n)} = \big\{\xi = (i_0, \ldots, i_n; t_1, \ldots, t_n) \in V^{n+1} \times \mathbb{R}^n \text{ with } t_1 < \ldots < t_n\big\}$$

equipped with the Borel σ-algebra and the product measure obtained from the counting measure on V^{n+1} and the Lebesgue measure on the n-simplex (normalized so that the volume of $(t_1 < \ldots < t_n)$ with $t_n < t$ is $t^n/n!$). We extend this to a finite measure ν_t on $\Xi_t = \bigcup_{n\in\mathbb{N}} \Xi_t^{(n)}$. We then let for all $\xi = (i_0, \ldots, i_n; t_1, \ldots, t_n)$ of $\bigcup_{t\in\mathbb{R}_+} \Xi_t$:

$$T_t(\xi) := \mathrm{e}^{(t-t_k)G_{i_k}}\, L_{i_k,i_{k-1}}\, \mathrm{e}^{(t_k-t_{k-1})G_{i_{k-1}}} \ldots \mathrm{e}^{(t_2-t_1)G_{i_1}}\, L_{i_1,i_0}\, \mathrm{e}^{t_1 G_{i_0}}. \tag{46}$$

if k is the greatest index such that $t_k \le t$, with $G_i = -\mathrm{i}H_i - \frac{1}{2}\sum_{j\in V} L_{j,i}^* L_{j,i}$. This $T_t(\xi)$ is an operator from $\mathfrak{h}_{i_0}$ to $\mathfrak{h}_{i_k}$, and

$$\int_{\Xi_t} T_t(\xi)\rho T_t(\xi)^*\, \mathrm{d}\nu_t(\xi) = \mathrm{e}^{t\mathcal{L}}(\rho) \tag{47}$$

where

$$\mathcal{L}(\rho) = -\mathrm{i}[H,\rho] + \sum_{i,j} \big(M_{i,j}\rho M_{i,j}^* - \frac{1}{2}M_{i,j}^* M_{i,j}\rho - \frac{1}{2}\rho M_{i,j}^* M_{i,j}\big),$$

with $H = \sum_i H_i \otimes |i\rangle\langle i|$ and $M_{i,j} = L_{i,j} \otimes |i\rangle\langle j|$, is the generator of a quantum dynamical semigroup (given in its Lindblad form, see [94]). The relation (47) is therefore the analogue of (3).

We then define for all state $\rho = \sum_{i\in V} \rho(i) \otimes |i\rangle\langle i|$ a probability measure on $\Xi := \bigcup_{t\in\mathbb{R}_+} \Xi_t$ by

$$\mathbb{P}_\rho(E) = \int_E \mathrm{tr}\,\big(T_t(\xi)\rho T_t(\xi)^*\big)\, \mathrm{d}\nu_t(\xi)$$

if $E \in \Xi_t$ (the fact that $\mathrm{e}^{t\mathcal{L}}$ preserves the trace ensures that the family thus defined is consistent, and this definition can be extended to Ξ). One then defines two stochastic processes $(v_t)_t$ and $(\varrho_t)_t$ on the probability space $(\Xi, \mathbb{P}_\rho)$ by letting for $\xi = (i_0, \ldots, i_n; t_1, \ldots, t_n)$

$$v_t(\xi) = \begin{cases} i_k & \text{si } t_k \le t < t_{k+1} \\ i_n & \text{si } t_n \le t \end{cases} \qquad \varrho_t(\xi) = \frac{T_t(\xi)\rho T_t(\xi)^*}{\mathrm{tr}\,\big(T_t(\xi)\rho(i_0)T_t(\xi)^*\big)}.$$

These processes define the continous time open quantum walk.

One can show that $\rho_t = \varrho_t \otimes |v_t\rangle\langle v_t|$ satisfies the stochastic differential equation

$$\mathrm{d}\rho_t = \mathcal{M}(\rho_{t-})\,\mathrm{d}t + \sum_{i,j} \big(\frac{M_{i,j}\rho_{t-}M_{i,j}^*}{\mathrm{tr}(M_{i,j}\rho_{t-}M_{i,j}^*)} - \rho_{t-}\big)\,\mathrm{d}N_t^{i,j} \tag{48}$$

where

$$\mathcal{M}(\rho) = \mathcal{L}(\rho) - \sum_{i,j} \big(M_{i,j}\,\rho\, M_{i,j}^* - \rho\, \mathrm{tr}(M_{i,j}\,\rho\, M_{i,j}^*)\big)$$

and $N_t^{i,j}$ is the process

$$N_t^{i,j}(i_0,\ldots,i_n;t_1,\ldots,t_n) = \mathrm{card}\,\big\{k=1,\ldots,n \text{ such that } t_k \le t \text{ et } (i_{k-1},i_k)=(i,j)\big\},$$

which one can prove (see [13]) has the same distribution as the process

$$\int_0^t \int_{\mathbb{R}} \mathbb{1}_{0<y<\mathrm{tr}(M_{i,j}\rho_{s-}M_{i,j}^*)}\, \tilde N^{i,j}(\mathrm{d}y,\mathrm{d}s)$$

if $\tilde N^{i,j}$ is an homogeneous Poisson process on $\mathbb{R}^2$. This shows that one can understand the trajectory-wise description above in the following way: if the "particle" is initially observed at i_0, then it evolves following $\varrho \rightsquigarrow \dfrac{\mathrm{e}^{tG_{i_0}}\varrho\mathrm{e}^{tG_{i_0}^*}}{\mathrm{tr}(\mathrm{e}^{tG_{i_0}}\varrho\mathrm{e}^{tG_{i_0}^*})}$ as long as it stays at position i_0; it will jump to position i_1 after an "exponential" time with intensity $\mathrm{tr}\,\big(L_{i_1,i_0}\rho_{s-}(i_0)L_{i_1,i_0}^*\big)$ (which therefore depends on the evolution of the internal degree of freedom), and its internal state will then undergo the evolution $\varrho \rightsquigarrow \dfrac{L_{i,j}\varrho L_{i,j}^*}{\mathrm{tr}(L_{i,j}\varrho L_{i,j}^*)}$. An element $\xi \in \Xi$ represents a possible trajectory of the particle, the times t_k indicating the jump times and the corresponding i_k indicating the destination of the jump. We refer the reader to section 1 of [P25] for technical details.

Remark 3.47. Because of the internal degrees of freedom, a continuous time OQW stopped at its sequence of jump times is not a discrete time OQW. Going from discrete time to continuous time is therefore a more delicate business with open quantum walks than with Markov chains.

This last remark forces us, in order to recover analogues of Theorems 3.21 and 3.23, to reconsider our proofs. Once the formalism is in place, however, proofs can be translated to the continous time, the effect $\varrho \rightsquigarrow L_\pi \varrho L_\pi^*$ of a trajectory π (see (9)) being replaced by the effect $\varrho \rightsquigarrow T_t(\xi)\varrho T_t(\xi)^*$ of a trajectory ξ. The techniques using concatenations of trajectories can be adapted and one obtains exact analogues of Theorems 3.21 and 3.23, for the return times T_i and the occupation times N_i, defined by

$$T_i = \inf\{t \ge \tau_1 | X_t = i\} \qquad N_i = \int_0^\infty \mathbb{1}_{v_t=i}\,\mathrm{d}t$$

(where τ_1 is the first jump time). We do not think it is necessary to detail these analogues.

5. Invariant measures for stochastic Schrödinger equations

The present section describes the results of article [P27], which extends those of [P24] to the continuous time setting. Here we wish to consider, quantum trajectories that involve both diffusive terms and jump terms. The construction of the model will be more complicated than for continuous time OQW; nevertheless, here also the proofs of the discrete time case will extend easily to continous time.

We consider once again a finite-dimensional Hilbert space $\mathcal{H}$, and consider a Hamiltonian H, and operators $L_i, i \in V$. Suppose that our set V of indices is finite, partition it into $V = V_b \cup V_p$, and consider $\big(\Omega, (\mathcal{F}_t)_t, \mathbb{P}\big)$ a filtered space satisfying the standard conditions, on which we have Brownian motions W_i

for $i \in V_b$, and standard Poisson processes N_j for $j \in V_p$, such that the family $\big(W_i, N_j;\ i \in V_b, j \in V_p\big)$ is independent. We define a process $(S_t)_t$ by

$$\mathrm{d}S_t = \big(K + \tfrac{\#V_p}{2}\,\mathrm{Id}\big)S_{t-}\,\mathrm{d}t + \sum_{i\in V_b} L_i S_{t-}\,\mathrm{d}W_i(t) + \sum_{j\in V_p} (L_j - \mathrm{Id})S_{t-}\,\mathrm{d}N_j(t), \tag{49}$$

by $S_0 = \mathrm{Id}$ (here $\#V_p$ is the cardinal of V_p), where $K = -\mathrm{i}H - \frac{1}{2}(\sum_{i\in V_b} L_i^* L_i + \sum_{j\in V_p} L_j^* L_j)$. The coefficients being Lipschitzian, this equation admits a unique strong solution. Let now $\rho \in \mathcal{S}_k$; we obtain easily from Itô calculus that

$$\mathbb{E}(S_t \rho S_t^*) = \mathrm{e}^{t\mathcal{L}}(\rho)$$

where

$$\mathcal{L}(\rho) = -\mathrm{i}[H,\rho] + \sum_{i\in V_b\cup V_p} L_i\rho L_i^* - \frac{1}{2}L_i^* L_i \rho - \frac{1}{2}\rho L_i^* L_i. \tag{50}$$

We then let

$$Z_{\rho,t} = \mathrm{tr}(S_t^* S_t \rho)$$

and

$$\rho_t = \frac{S_t \rho S_t^*}{\mathrm{tr}(S_t\rho S_t^*)}$$

if $Z_{\rho,t} \neq 0$ (below, as was the case in section 3, this will be almost surely verified). One can then show that $(Z_{\rho,t})_t$ is a martingale for $\mathbb{P}$, and define a new probability measure $\mathbb{P}_\rho$ by

$$\mathrm{d}\mathbb{P}_\rho|_{\mathcal{F}_t} = Z_{\rho,t}\,\mathrm{d}\mathbb{P}|_{\mathcal{F}_t}.$$

An application of Girsanov's theorem then tells us that if one lets

$$B_i^\rho(t) = W_i(t) - \int_0^t \mathrm{tr}\left((L_i + L_i^*)\rho_{s-}\right)\mathrm{d}s,$$

then under $\mathbb{P}_\rho$, the B_i^ρ, $i \in I_b$, are independent Wiener processes, and each N_j is a point process of intensity $t \mapsto \mathrm{tr}(C_j^* C_j \rho_{t-})$. In addition, $(\rho_t)_t$ satisfies

$$\begin{aligned}\mathrm{d}\rho_t =& \mathcal{L}(\rho_{t-})\mathrm{d}t\\ &+ \sum_{i=0}^{\infty}\Big(L_i\rho_{t-} + \rho_{t-}L_i^* - \mathrm{tr}\left(\rho_{t-}(L_i+L_i^*)\right)\rho_{t-}\Big)\mathrm{d}B_i^\rho(t)\\ &+ \sum_{j=0}^{\infty}\Big(\frac{L_j\rho_{t-}L_j^*}{\mathrm{tr}(L_j\rho_{t-}L_j^*)} - \rho_{t-}\Big)\Big(\mathrm{d}N_j(t) - \mathrm{tr}(L_j\rho_{t-}L_j^*)\,\mathrm{d}t\Big).\end{aligned} \tag{51}$$

We have in addition

$$\mathbb{E}_\rho(\rho_t) = \mathrm{e}^{t\mathcal{L}}(\rho).$$

As in the discrete case, if ρ_0 is a pure state, i.e. is of the form $\rho_0 = |x_0\rangle\langle x_0|$, then for all t, $\rho_t = |\hat{x}_t\rangle\langle\hat{x}_t|$, with $x_t = \frac{S_t x_0}{\|S_t x_0\|}$.

Remark 3.48. This property, together with relation (3), is the starting point for Monte-Carlo wave functions technique for simulation of solutions of master equations, see [40].

A continuous-time version of Theorem 3.3 (which can be found for example in [95]) shows once more than an invariant measure for ρ_t will have support in the set of pure states, and we therefore focus on the process $(\hat{x}_t)_t$ on $\mathrm{P}\mathcal{H}$. One can then give a similar statement to that of Theorem 3.39. For this it is necessary to adapt condition **(Pur)**. A direct translation of that condition would be that any orthogonal projector π such that $\pi S_t^* S_t \pi \propto \pi$ $\mathbb{P}$-almost surely for all $t \geq 0$ is of rank 0 or 1. However, this condition is difficult to check in practice. We therefore replace it with the following sufficient condition:

(cPur) Any orthogonal projector π such that for all $i \in V_b$, $\pi(L_i + L_i^*)\pi \propto \pi$ and for all $j \in V_p$, $\pi L_j^* L_j \pi \propto \pi$, is of rank 0 or 1.

We then have the following result:

THEOREM 3.49 ([P27]). *Suppose that the family $(L_i)_{i \in V_b \cup V_p}$ satisfies assumption* **(cPur)** *and that* 0 *is a simple eigenvalue of* $\mathcal{L}$ *defined by* (50). *Then the Markov chain* $(\hat{x}_t)_t$ *admits a unique invariant probability measure* ν_{inv} *and there exist* $C > 0$ *and* $\tau > 0$ *such that for all initial distribution* ν*, the distribution* ν_t *of* $\hat{x}_t$ *satisfies*

$$W_1(\nu_t, \nu_{\mathrm{inv}}) \leq C\mathrm{e}^{-\tau t}.$$

The proof is essentially identical to that of Theorem 3.39, exploiting the fact that the process

$$M_t = \frac{S_t^* S_t}{\mathrm{tr}(S_t^* S_t)}$$

is a martingale under $\mathbb{P}_{\mathrm{ch}}$.

Remark 3.50. The article [14] of Barchielli and Paganoni had already shown that condition **(cPur)** implied the purification of $(\rho_t)_t$. The same article gave sufficient conditions for the uniqueness of the invariant mesure in the diffusive case (i.e. when equations (49) or (51) do not contain any jump term), but these conditions were hard to check and therefore not satisfactory.

Chapter 4

Time-dependent systems: the adiabatic case

In the present chapter, we study situations in which the considered instrument depends on time. Our initial motivation comes from the study of Landauer's principle for repeated interactions systems, in the adiabatic case. We will first explain all three elements of this last sentence: Landauer's principle, repeated interactions systems, and adiabatic evolutions.

As we wrote in 0.4, Landauer's principle can be summarized as follows: following an irreversible transformation of the state of a system $\mathcal{S}$ through its interaction with an environment $\mathcal{E}$ initially at thermal equilibrium at inverse temperature β, one must have

$$\beta \Delta Q_{\mathcal{E}} \geq \Delta S_{\mathcal{S}} \tag{1}$$

where $\Delta Q_{\mathcal{E}}$ is the variation of free energy of $\mathcal{E}$ and $\Delta S_{\mathcal{S}}$ the variation of entropy of $\mathcal{S}$. In addition, one expects the inequality (1) to saturate when the evolution is adiabatic, i.e. infinitely slow, and starting from a joint system $\mathcal{S} \vee \mathcal{E}$ initially at equilibrium (see below). The surprising diversity of the commentaries on Landauer's principle can probably be traced back to the absence of a proof of (1), or even to the absence of a precise statement (note for instance that we have yet to define precisely the quantities $\Delta Q_{\mathcal{E}}$ and $\Delta S_{\mathcal{S}}$). The first sound proof, due to Reeb and Wolf in [111], is expressed in the quantum formalism, which is hardly surprising considering the order of magnitude of the involved quantities[1]. To state it, suppose that the system $\mathcal{S}$ (respectively the environment $\mathcal{E}$) is described by a Hilbert space $\mathcal{H}_{\mathcal{S}}$ (resp. $\mathcal{H}_{\mathcal{E}}$) of finite dimension equipped with a Hamiltonian $H_{\mathcal{S}}$ (resp. $H_{\mathcal{E}}$), and a state ρ^{i} (resp. ξ). We suppose that the state ξ is a Gibbs state at inverse temperature β, that is, $\xi = \exp(-\beta H_{\mathcal{E}})/Z$ with $Z = \operatorname{tr}\big(\exp(-\beta H_{\mathcal{E}})\big)$. We then couple $\mathcal{S}$ and $\mathcal{E}$ via the unitary $U \in \mathcal{B}(\mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{E}})$, then decouple them after the interaction. The system $\mathcal{S}$ is then in the state

$$\rho^{\mathrm{f}} = \operatorname{tr}_{\mathcal{E}}\big(U(\rho^{\mathrm{i}} \otimes \xi)U^*\big). \tag{2}$$

We define

$$\Delta S_{\mathcal{S}} := S(\rho^{\mathrm{i}}) - S(\rho^{\mathrm{f}}), \qquad \Delta Q_{\mathcal{E}} := \operatorname{tr}\big(U(\rho^{\mathrm{i}} \otimes \xi)U^* H_{\mathcal{E}}\big) - \operatorname{tr}\big((\rho^{\mathrm{i}} \otimes \xi)H_{\mathcal{E}}\big),$$

the entropy decrease of the system and the energy increase of the environment. One then has by a short and direct calculation (see [111] or [P19]) the *entropy balance equation*:

$$\beta \Delta Q_{\mathcal{E}} = \Delta S_{\mathcal{S}} + \sigma \tag{3}$$

[1] For example, the energetic cost of the resetting (to a known pure state) of a bit of information would have a minimal cost, at room temperature, of approximately 3.10^{-21} joules; according to the literature, the current technolologies should reach this bound by 2035, see [34].

where

$$\sigma = S\big(U(\rho^{\rm i}\otimes\xi)U^*\,|\,\rho^{\rm f}\otimes\xi\big). \tag{4}$$

Since a relative entropy is always nonnegative, the equality (3) implies the Landauer bound (1). This simple and clear proof has the disadvantage of assuming that the environment is described by a Hilbert space $\mathcal{H}_\mathcal{E}$ of finite dimension; the article [79] then extended the above approach to the case where the environment is described by a general C*-algebra, a necessary step for the discussion of the saturation of the inequality (see below). The article [P19] considers the same problem when the environment is described by a repeated interactions system.

A *repeated interactions systems* (RIS) is composed of a fixed system $\mathcal{S}$, which interacts for a given time with a system $\mathcal{E}_1$, following a Hamiltonian dynamics, before the coupling of $\mathcal{S}$ and $\mathcal{E}_1$ is turned off; everything therefore happens as we described above, except that after this first step, $\mathcal{S}$ interacts with another system $\mathcal{E}_2$, and so on. The parameters describing the k-th interaction are $H_{\mathcal{E}_k}$, β_k and v_k, these parameters determining the unitary

$$U_k := \exp\big(-\mathrm{i}\tau(H_\mathcal{S}\otimes \mathrm{Id} + \mathrm{Id}\otimes H_{\mathcal{E}_k} + v_k)\big)$$

(we therefore assume for notational simplicity that $\mathcal{H}_{\mathcal{E}_k}\equiv\mathcal{H}_\mathcal{E}$ and that the interaction times are constant, and equal to τ). One has in particular $\xi_k = \mathrm{e}^{-\beta_k H_{\mathcal{E}_k}}/\operatorname{tr}(\mathrm{e}^{-\beta_k H_{\mathcal{E}_k}})$. It is clear that this model is strongly connected to models of indirect measurements of chapter 3. The RIS are also used to model the interaction of $\mathcal{S}$ with an environment, which here consists of the sequence $\mathcal{E}_1$, $\mathcal{E}_2$, etc., which has by definition two of the expected characteristics of an environment: an infinite total energy, and (at least when all the states ξ_k are identical) a characteristic time of return to equilibrium small before that of $\mathcal{S}$ (which is by definition egal à τ). The interest for RIS as model for an environment was stimulated (on the mathematical side) by the articles [P2], [9] but of course most of all by physical experiments such as those of cavity quantum electrodynamics; we refer the reader to the review [31] for more information. Consider therefore a RIS; a direct application of (3) gives

$$\Delta S_\mathcal{E}^{\rm tot} = \Delta S_\mathcal{S}^{\rm tot} + \sigma_T^{\rm tot} \tag{5}$$

where

$$\Delta S_\mathcal{E}^{\rm tot} := \sum_{k=1}^T \beta_k \Delta Q_{\mathcal{E}_k} \qquad \Delta S_\mathcal{S}^{\rm tot} := \sum_{k=1}^T \Delta S_{k,\mathcal{S}} \qquad \sigma_T^{\rm tot} := \sum_{k=1}^T \sigma_k.$$

Note that the quantity $\beta_k\Delta Q_{\mathcal{E}_k}$ is a Clausius entropy, which justifies our notation $\Delta S_\mathcal{E}^{\rm tot}$. One has trivially from the earlier discussion $\sigma^{\rm tot}\geq 0$.

Last, the adiabatic regime is the situation in which one considers in large time $T\to\infty$ a dynamics with slowly varying parameters – typically by dilating to $[0,T]$ a family of parameters that are regular functions on $[0,1]$. The physical relevance of this limit comes from the adiabatic "Theorem" of Born and Fock (in [23]) that says that, if the dynamics is Hamiltonian, then the evolution between times 0 and T resulting from this slow deformation of parameters has the property of sending any eigenstate of the initial Hamiltonian to an eigenstate of the final Hamiltonian. As we wrote before, one expects a saturation of the inequality (1) in the adiabatic limit. Yet one can prove (see [79]) that in the finite-dimensional framework of [111], one has equality in (1) if and only if $\Delta S_\mathcal{S} = \Delta Q_\mathcal{E} = 0$, in which case $\rho^{\rm i}$ and $\rho^{\rm f}$ are unitarily equivalent, and $\xi = \operatorname{tr}_\mathcal{E}\big(U(\rho^{\rm i}\otimes\xi)U^*\big)$. As a consequence, in this setup, the saturation of (1) holds only in trivial cases. In the C*-algebraic framework, the article [79] shows the saturation of (1), and therefore the cancellation of σ, in the adiabatic limit (under assumptions of stability of deformed dynamical systems).

Our goal was therefore initially to study the behaviour of σ_T^{tot} for repeated interactions system, when $T \to \infty$ in the adiabatic limit. The corresponding results are described in [P19] (and are given here in remarks 4.17 and 4.19). They appear below as corollaries of more precise results obtained in [P22], and are motivated again by two-time measurements and the aim to give statistical formulations of the laws of thermodynamics. Indeed, Landauer's principle as stated above concerns $\Delta Q_{\mathcal{E}}$ and $\Delta S_{\mathcal{S}}$, which are average quantities. In the context of the program which we have started in chapter 2, one can hope to refine this principle to a relation between the distributions of random variables corresponding to the variation of energy, or entropy, along a trajectory. Such a result was obtained in [19] in a Hamiltonian setup with a reservoir modeled by a C*-algebra. We wanted to study the analogous question for repeated interactions systems. Let us therefore begin by setting up the formalism of two-time measurements in this setup.

1. Two-time measurements of repeated interactions systems

Assume that we have a repeated interactions system for times going from $k = 1$ to T, described by parameters $H_{\mathcal{E}_k}$, β_k, v_k and consequently $\xi_k = \mathrm{e}^{-\beta_k H_{\mathcal{E}_k}} / \operatorname{tr}(\mathrm{e}^{-\beta_k H_{\mathcal{E}_k}})$. We will denote by Ξ the product $\xi_1 \otimes \ldots \otimes \xi_T$. We define on these objects two experimental protocols, that depend on the choice of observables A^{i} and A^{f} of $\mathcal{S}$, and T observables $M_1,\ldots,M_T$ of $\mathcal{E}_1,\ldots,\mathcal{E}_T$ respectively.

The first protocol, called "forward protocol" is defined as follows:

1. measure the observable A^{i} of $\mathcal{S}$, and denote by a^{i} the outcome;
2. at each step $k = 1 \nearrow T$:
    a. measure the observable M_k of $\mathcal{E}_k$, and denote by i_k the outcome;
    b. let the joint system $\mathcal{S} \vee \mathcal{E}_k$ evolve following the unitary U_k;
    c. measure again M_k, and denote by j_k the outcome;
3. measure the observable A^{f} of $\mathcal{S}$, and denote by a^{f} the outcome.

The full list of measurement outcomes can therefore be written as an element

$$(a^{\mathrm{i}}, \vec\imath, \vec\jmath, a^{\mathrm{f}}) := (a^{\mathrm{i}}; (i_1, j_1), \ldots, (i_T, j_T); a^{\mathrm{f}})$$

of

$$\Omega_T = \mathrm{sp}\, A^{\mathrm{i}} \times (\mathrm{sp}\, M_1)^2 \times \ldots \times (\mathrm{sp}\, M_T)^2 \times \mathrm{sp}\, A^{\mathrm{f}},$$

and according to the Born rule, the probability associated with ω in this "forward" experiment is

$$\mathbb{P}_T^F(a^{\mathrm{i}}, \vec\imath, \vec\jmath, a^{\mathrm{f}}) := \operatorname{tr}\big(U_T \ldots U_1 (\pi^{\mathrm{i}}_{a^{\mathrm{i}}} \otimes \Pi_{\vec\imath})(\rho^{\mathrm{i}} \otimes \Xi)(\pi^{\mathrm{i}}_{a^{\mathrm{i}}} \otimes \Pi_{\vec\imath}) U_1^* \ldots U_T^* (\pi^{\mathrm{f}}_{a^{\mathrm{f}}} \otimes \Pi_{\vec\jmath})\big), \qquad (6)$$

where $\pi^{\mathrm{i}}_{a^{\mathrm{i}}}$ (respectively $\pi^{\mathrm{f}}_{a^{\mathrm{f}}}$) is the spectral projector of A^{i} (resp. A^{f}) associated with the eigenvalue a^{i} (resp. a^{f}), and $\Pi_{\vec\imath}$ is the product $\Pi^{(1)}_{i_1} \otimes \ldots \otimes \Pi^{(T)}_{i_T}$ of the spectral projectors of $M_1 \otimes \ldots \otimes M_T$ associated with $\vec\imath = (i_1, \ldots, i_T)$ (and similarly for $\Pi_{\vec\jmath}$).

The second protocol, called "backward protocol" is defined in the following way:

1. measure the observable A^{f} of $\mathcal{S}$, and denote by a^{f} the outcome;

2. at each step $k = T \searrow 1$:

   a. measure the observable M_k of $\mathcal{E}_k$, and denote by j_k the outcome;

   b. let the joint system $\mathcal{S} \vee \mathcal{E}_k$ evolve following the unitary U_k^*;

   c. measure again M_k, and denote by i_k the outcome;

3. measure the observable A^{i} of $\mathcal{S}$, and denote by a^{i} the outcome.

The probability associated with the results $(a^{\mathrm{i}}; (i_1, j_1), \dots, (i_T, j_T); a^{\mathrm{f}})$ (where the order of indices no longer indicates the chronological order of the measurements) is then

$$\mathbb{P}_T^B(a^{\mathrm{i}}, \vec{\imath}, \vec{\jmath}, a^{\mathrm{f}}) := \mathrm{tr}\left(U_1^* \dots U_T^*(\pi_{a^{\mathrm{f}}}^{\mathrm{f}} \otimes \Pi_{\vec{\jmath}})(\rho_T^{\mathrm{f}} \otimes \Xi)(\pi_{a^{\mathrm{f}}}^{\mathrm{f}} \otimes \Pi_{\vec{\jmath}})U_T \dots U_1(\pi_{a^{\mathrm{i}}}^{\mathrm{i}} \otimes \Pi_{\vec{\imath}})\right). \tag{7}$$

We then define three random variables on Ω_T. The first two are natural:

$$\Delta a_T(a^{\mathrm{i}}, \vec{\imath}, \vec{\jmath}, a^{\mathrm{f}}) := a^{\mathrm{i}} - a^{\mathrm{f}}, \tag{8}$$

$$\Delta m_T^{\mathrm{tot}}(a^{\mathrm{i}}, \vec{\imath}, \vec{\jmath}, a^{\mathrm{f}}) := \sum_{k=1}^{T}(y_{j_k}^{(k)} - y_{i_k}^{(k)}). \tag{9}$$

Remark 4.1.

1. The definition of Δa_T as $a^{\mathrm{i}} - a^{\mathrm{f}}$, and therefore as a decrease of the quantity a, is consistent with the definition of $\Delta S_{\mathcal{S}}$ above, which follows the usage concerning Landauer's principle.

2. In the case of time-dependent systems, because of the dependency in T of $U_k = U(\frac{k}{T})$ the family $(\mathbb{P}_T^F)_T$ is not in general consistent; it therefore does not define a probability $\mathbb{P}^F$ on the set of infinite outcome sequences, as was the case in chapter 3.

3. If one ignores the measurement steps 2.a. and 2.c. of the forward protocol, then the resulting protocol corresponds to T steps of the situation defined on page 81 at the beginning of this chapter, except that one applies at step k the unitary U_k coupling $\mathcal{S}$ and $\mathcal{E}_k$. Then the state of $\mathcal{S}$ the k-th step of this protocol without measurements of the probe is

   $$\rho_k = \mathrm{tr}_{\mathcal{E}}\left(U_k(\rho_{k-1} \otimes \xi_k^{\mathrm{i}})U_k^*\right)$$

   (we denote $\mathrm{tr}_{\mathcal{E}}$ in place of $\mathrm{tr}_{\mathcal{H}_{\mathcal{E}}}$ to simplify the notation), which can be written as $\rho_k = \mathcal{L}_k(\rho_{k-1})$, where $\mathcal{L}_k$ is a quantum channel. We then have

   $$\rho_k = \mathcal{L}_k \circ \dots \circ \mathcal{L}_1(\rho^{\mathrm{i}}).$$

We suppose in the sequel that the states ρ^{i} and ρ_T^{f} are faitfhul (the faithfulness of ρ_T^{f} will follow from that of ρ^{i}). This ensures that

$$\mathbb{P}_T^F(a^{\mathrm{i}}, \vec{\imath}, \vec{\jmath}, a^{\mathrm{f}}) = 0 \text{ if and only if } \mathbb{P}_T^B(a^{\mathrm{i}}, \vec{\imath}, \vec{\jmath}, a^{\mathrm{f}}) = 0. \tag{10}$$

The property (10) plays the role of the assumption **(B)** of chapter 3, and allows to define a third random variable:

$$\varsigma_T(\omega) := \log \frac{\mathbb{P}_T^F(a^{\mathrm{i}}, \vec{\imath}, \vec{\jmath}, a^{\mathrm{f}})}{\mathbb{P}_T^B(a^{\mathrm{i}}, \vec{\imath}, \vec{\jmath}, a^{\mathrm{f}})},$$

which we call the entropy production along the trajectory $\omega = (a^{\mathrm{i}}, \vec{\imath}, \vec{\jmath}, a^{\mathrm{f}})$. This random variable analogous to ς_n of section 3.2 can once again be seen as a log-likelihood.

We suppose from now on that A^{i}, A^{f} and the M_k are respectively functions of ρ^{i}, ρ^{f} and ξ_k (or equivalently of $H_{\mathcal{E}_k}$). We then have the following result, obtained by a direct computation:

Lemma 4.2 ([P22]). *We have the identity*

$$\varsigma_T(\omega) = \log\Big(\frac{\mathrm{tr}(\rho^{\mathrm{i}}\pi^{\mathrm{i}}_{a^{\mathrm{i}}})}{\mathrm{tr}(\rho^{\mathrm{f}}_T\pi^{\mathrm{f}}_{a^{\mathrm{f}}})}\,\frac{\dim \pi^{\mathrm{f}}_{a^{\mathrm{f}}}}{\dim \pi^{\mathrm{i}}_{a^{\mathrm{i}}}}\Big) + \sum_{k=1}^{T}\beta_k(E^{(k)}_{j_k} - E^{(k)}_{i_k}), \tag{11}$$

where the $E^{(k)}_{i_k}$ are the eigenvalues (repeated with multiplicity) of $H_{\mathcal{E}_k}$.

This calculation is particularly relevant when the measurement observables of the system $\mathcal{S}$ are entropy observables:

Proposition 4.3 ([P22]). *Assume $A_i = -\log\rho^{\mathrm{i}}$, $A_f = -\log\rho^{\mathrm{f}}$ and $M(s) = \beta(s)H_{\mathcal{E}}(s)$ (or $M(s) = -\log\xi(s)$). We then have if $\rho^{\mathrm{i}} = \sum r^{\mathrm{i}}_a\pi^{\mathrm{i}}_a$ and $\rho^{\mathrm{f}}_T = \sum r^{\mathrm{f}}_a\pi^{\mathrm{f}}_a$:*

$$\varsigma_T(\omega) = (-\log r^{\mathrm{f}}_{a^{\mathrm{f}}}) - (-\log r^{\mathrm{i}}_{a^{\mathrm{i}}}) + \sum_{k=1}^{T}\beta_k(E^{(k)}_{j_k} - E^{(k)}_{i_k}) = -\Delta a_T + \Delta m^{\mathrm{tot}}_T. \tag{12}$$

In addition, denoting $\mathbb{E}_T$ the expectation with respect to $\mathbb{P}^F_T$, we have

$$\mathbb{E}_T(\Delta a_T) = S(\rho^{\mathrm{i}}) - S(\rho^{\mathrm{f}}) = \Delta S^{\mathrm{tot}}_{\mathcal{S}} \qquad \mathbb{E}_T(\Delta m^{\mathrm{tot}}_T) = \sum_{k=1}^{T}\beta_k\Delta Q_k = \Delta S^{\mathrm{tot}}_{\mathcal{E}} \tag{13}$$

and as a consequence

$$\mathbb{E}_T(\varsigma_T) = \sigma^{\mathrm{tot}}_T. \tag{14}$$

The identity (12) gives an entropy balance equation at the level of trajectories, and follows directly from (11). If the initial and final measurements of $\mathcal{S}$ are suppressed from the protocol, then (12) reduces to a quantum version of the Tasaki–Crooks formula (see [37]). The relations (13) are obtained by a direct computation, and give with (3) the identity (14).

Remark 4.4. The idea of defining the entropy production along a trajectory as the difference of log-likelihoods of a trajectory and the reverse trajectory is standard (and goes back to [58]); however, there remains in a case like ours to make a choice concerning the definition of the reverse trajectory (see [77]). In particular, the choice made here (the reverse trajectory is the one obtained by the "backward" protocol) is not the same as that of example 3.36 originating from [18], which applies the theory from [P20]. Our choice of definition for the entropy production of a trajectory has therefore the advantage of having as expectation the average entropy production σ^{tot} (see also [3] for a justification of the initial and final measurements on $\mathcal{S}$), and of leading to the satisfactory results given below. One can also remark that, if one considers a time-independent RIS (i.e. one where the functions $H_{\mathcal{E}}, \beta, v$ are constant) a faithful and $\mathcal{L}$-invariant initial state $\rho = \rho^{\mathrm{i}}$, then $\rho^{\mathrm{f}} = \rho^{\mathrm{i}}$ and with the notation of section 3.2, we have

$$\mathbb{P}_\rho\big((i_1,j_1),\dots,(i_T,j_T)\big) = \sum_{a,b}\mathbb{P}^F_T(a,\vec{\imath},\vec{\jmath},b) = \mathrm{tr}\,\big(U_T\dots U_1\,(\rho\otimes\,\Xi\,\Pi_{\vec{\imath}})\,U_1^*\dots U_T^*(\mathrm{Id}\otimes\Pi_{\vec{\jmath}})\big)$$

but in general $\sum_{a,b} \mathbb{P}_T^B(a,\vec{\imath},\vec{\jmath},b) \neq \widehat{\mathbb{P}}_\rho\big((i_1,j_1),\ldots,(i_T,j_T)\big)$ because

$$\sum_{a,b} \mathbb{P}_T^B(a,\vec{\imath},\vec{\jmath},b) = \operatorname{tr}\big(U_1^*\ldots U_T^*\,(\rho\otimes\Xi\,\Pi_{\vec{\jmath}})\,U_T\ldots U_1(\mathrm{Id}\otimes\Pi_{\vec{\imath}})\big)$$

$$\widehat{\mathbb{P}}_\rho\big((i_1,j_1),\ldots,(i_T,j_T)\big) = \operatorname{tr}\big(U_1\ldots U_T\,(\rho\otimes\,\Xi\,\Pi_{\vec{\jmath}})\,U_T^*\ldots U_1^*(\mathrm{Id}\otimes\Pi_{\vec{\imath}})\big).$$

On the other hand, under the assumption of time reversal invariance described in (20) (an assumption which is made in [18]), these two expressions coincide. In that case, the protocols related to example 3.36 therefore amount to suppressing measurements on $\mathcal{S}$ in our forward and backward protocols.

We now wish to investigate the distributions of the variables Δa_T, $\Delta m_T^{\mathrm{tot}}$ and ς_T. Once again, the most practical tool is the moment generating function of these distributions.

Proposition 4.5 ([P22]). *Assume that A^{i}, A^{f} and the M_k are functions of ρ^{i}, ρ^{f} and ξ_k. Then the generating function of $(\Delta m_T^{\mathrm{tot}}, \Delta a_T)$ can be expressed as*

$$\Gamma_{(\Delta m_T^{\mathrm{tot}},\Delta a_T)}(\alpha_1,\alpha_2) = \operatorname{tr}\big(\mathrm{e}^{-\alpha_2 A^{\mathrm{f}}}\mathcal{L}_M^{(\alpha_1)}[T]\ldots\mathcal{L}_M^{(\alpha_1)}[1](\mathrm{e}^{+\alpha_2 A^{\mathrm{i}}}\rho^{\mathrm{i}})\big) \tag{15}$$

where $\mathcal{L}_M^{(\alpha)}[k]$ is a completely positive map which admits a Kraus decomposition with operators $K_{i,j}^{(\alpha)}[k]$, $i,j\in\{1,\ldots,\dim\mathcal{H}_\mathcal{E}\}$:

$$K_{i,j}^{(\alpha)}[k] = \mathrm{e}^{\frac{\alpha}{2}(m_j[k]-m_i[k])}K_{i,j}[k]$$

where $K_{i,j}[k]$ is defined by

$$\langle x, K_{i,j}[k]\,y\rangle_{\mathcal{H}_\mathcal{S}} = \langle x\otimes\mu_j[k], U_k\,y\otimes\xi_k^{1/2}\mu_i[k]\rangle$$

if $\mu_i[k]$, $i=1,\ldots,\dim\mathcal{H}_\mathcal{E}$ is an orthonormal basis of eigenvectors of M_k associated with the eigenvalues $m_i[k]$, $i=1,\ldots,\dim\mathcal{H}_\mathcal{E}$.

Assume for a moment that our RIS does not depend on time; in that case the asymptotic behaviour of $\frac{1}{T}\log\Gamma_{(\Delta m_T^{\mathrm{tot}},\Delta a_T)}(\alpha_1,\alpha_2)$ in the limit $T\to\infty$ can be studied in the same way as we did to obtain (3.11). Expression (15) of that quantity, which contains a product of the successive values of $\mathcal{L}_M^{(\alpha)}[k]$, is an indication that an adiabatic theorem could help us express this asymptotics if the $\mathcal{L}^{(\alpha)}[k]$ were obtained by sampling a sufficiently regular function, i.e. if we had $\mathcal{L}_M^{(\alpha)}[k] = \mathcal{L}_M^{(\alpha)}(\frac{k}{T})$. We will assume from now on that this is the case, by considering adiabatic repeated interactions systems (ARIS).

ARIS An ARIS is a family of RIS indexed by an "adiabatic parameter" $T\in\mathbb{N}^*$, such that there exist twice continously differentiable functions[2] $s\mapsto H_\mathcal{E}(s)$, $\beta(s)$, $v(s)$ on $[0,1]$ such that for every value of the "adiabatic parameter" $T\in\mathbb{N}^*$ one has

$$H_{\mathcal{E}_k} = H_\mathcal{E}(\tfrac{k}{T}),\qquad \beta_k = \beta(\tfrac{k}{T}),\qquad v_k = v(\tfrac{k}{T})\qquad \text{for all } k=1,\ldots,T.$$

We denote consequently $\xi(s) = \mathrm{e}^{-\beta(s)H_\mathcal{E}(s)}/\operatorname{tr}(\mathrm{e}^{-\beta(s)H_\mathcal{E}(s)})$ and $\mathcal{L}_M^{(\alpha)}(s)$ a completely positive map with Kraus decomposition given by the $K_{i,j}^{(\alpha)}(s) = \mathrm{e}^{\frac{\alpha}{2}(m_j(s)-m_i(s))}K_{i,j}(s)$ where

$$\langle x, K_{i,j}(s)\,y\rangle_{\mathcal{H}_\mathcal{S}} = \langle x\otimes\mu_j(s), U_k\,y\otimes\xi_k^{1/2}\mu_i(s)\rangle$$

if the $\mu_i(s)$ form an orthonormal basis of eigenvalues of $M(s)$ associated with $m_i(s)$. In other words, $\mathcal{L}_M^{(\alpha)}(s)$ is defined in such a way that $\mathcal{L}_M^{(\alpha)}[k] = \mathcal{L}_M^{(\alpha)}(\frac{k}{T})$.

[2] We will say that a function f is C^2 on $[0,1]$ if it is C^2 on $(0,1)$, and that f' and f'' have finite limits at $0^+, 1^-$.

2. Discrete non unitary adiabatic theorem

In this section, we develop an adiabatic theorem that applies to the product $\mathcal{L}_M^{(\alpha_1)}(\frac{T}{T})\ldots\mathcal{L}_M^{(\alpha_1)}(\frac{1}{T})$ which appears in (15). Let us begin by considering the case $\alpha = 0$, in which case the $\mathcal{L}_M(\frac{k}{T})$ are quantum channels, and therefore contractions. Unfortunately, the standard adiabatic theorems apply to continous families of unitaries (see [82]), and even though there existed in the literature adiabatic theorems for continuous families of contractions (see [11, 12]), or to discrete families of unitaries (see [119]), there was no such theorem for discrete families of contractions. We therefore developed in [P19], a first adiabatic theorem that would apply to discrete families of contractions such as $\big(\mathcal{L}_M(\frac{k}{T})\big)_k$. This theorem however only allowed us to treat the cass $\alpha = 0$, and therefore restricted the results of [P19] to the level of expectations, i.e. of the average quantities $\Delta S_{\mathcal{S}}$, $\Delta S_{\mathcal{E}}$ and σ_T^{tot}. An improved adiabatic theorem, that replaced some assumptions regarding norms by spectral assumptions (and therefore less sensitive to perturbations) that applied to $(\mathcal{L}_M^{(\alpha)}(\frac{k}{T}))_k$, was then developed in [P22]; it is that newer theorem that we present below.

We consider a family $(F(s))_{s\in[0,1]}$ of linear maps one has Banach space B, that verifies the following conditions:

- **(a1)** the map $s \mapsto F(s)$ from $[0,1]$ to $\mathcal{B}(X)$ is continuous;
- **(a2)** for all $s \in [0,1]$, the spectral radius $\operatorname{spr} F(s)$ of $F(s)$ is equal to 1;
- **(a3)** the peripheral spectrum of $F(s)$ consists for all $s \in [0,1]$ of a constant finite number z of semi-simple eigenvalues;
- **(a4)** if we denote by $P(s)$ the spectral projector of $F(s)$ associated with the set of peripheral eigenvalues, the map $s \mapsto F^P(s) := F(s)P(s)$ is twice continuously differentiable from $[0,1]$ to $\mathcal{B}(X)$;
- **(a5)** if we denote $Q(s) := \mathrm{Id} - P(s)$, then

$$\ell := \sup_{s\in[0,1]} \operatorname{spr} F(s)Q(s) < 1.$$

A family satisfying all of the above assumptions will be called *admissible*. The assumptions **(a3)**, **(a4)** and standard perturbation theory (see [83]) ensure that one can parameterize the peripheral eigenvalues $\lambda^{(m)}(s)$ and the associated spectral projectors $P^{(m)}(s)$ in such a way that $s \mapsto \lambda^{(m)}(s)$ and $s \mapsto P^{(m)}(s)$ are C^2. One can then define a family of operators relevant to the statement our adiabatic theorem: we denote by $W : [0,1] \to \mathcal{B}(X)$ the unique family satisfying

$$W'(s) = \sum_{m=1}^{z} \frac{\mathrm{d}}{\mathrm{d}s}P^{(m)}(s)\, P^{(m)}(s)\, W(s),\ W(0) = \mathrm{Id}. \tag{16}$$

The $W(s)$ then verify the intertwining property $W(s)P^{(m)}(0) = P^{(m)}(s)W(s)$ (see section II.5 of [83]). We can now state our theorem:

THEOREM 4.6 ([P22]). *If the family* $\big(F(s)\big)_{s\in[0,1]}$ *is admissible, then for all* $\ell' \in (\ell,1)$ *there exist* $C>0$ *and* $T_0\in\mathbb{N}$ *such that for* $T\geq T_0$ *one has*

$$\sup_{k=1,\dots,T}\big\|F(\tfrac{k}{T})\dots F(\tfrac{1}{T}) - \sum_{m=1}^{z}\big(\prod_{n=1}^{k}\lambda^{(m)}(\tfrac{n}{T})\big)\,W(\tfrac{k}{T})P^{(m)}(0) - F^Q(\tfrac{k}{T})\dots F^Q(\tfrac{1}{T})Q(0)\big\| \\ \leq \frac{C}{T(1-\ell')}$$

for all $T\geq T_0$*, where* $F^Q(s):=Q(s)F(s)$*. In addition,*

$$\|F^Q(\tfrac{k}{T})\dots F^Q(\tfrac{1}{T})Q(0)\| \leq C{\ell'}^k,$$

where C *depends only on*

$$C_P := \sup_{s\in[0,1]}\ \sup_{m=1,\dots,z}\ \max\big(\|P^m(s)\|, \|P^{m\prime}(s)\|\big)$$

in a continuous manner, and T_0 *depends only on* C_P *and* ℓ'*.*

We will show in section 3 that the peripheral eigenvalues of $\mathcal{L}^{(\alpha)}$ are simple; this is the reason why we consider a version of Theorem 4.6 specific to that situation.

COROLLARY 4.7 ([P22]). *If the family* $\big(F(s)\big)_{s\in[0,1]}$ *is admissible, and its peripheral spectrum is simple, with the eigenvalue* $\lambda^{(m)}$ *having spectral projector* $P^{(m)}(s)=\phi_m(s)\psi_m^*(s)$*, then for all* $\ell'\in(\ell,1)$*, there exist* $C>0$ *and* $T_0\in\mathbb{N}$*, such that for* $T\geq T_0$ *and* $k\leq T$*,*

$$\big\|F(\tfrac{k}{T})\dots F(\tfrac{1}{T}) - \sum_{m=1}^{z}\big(\prod_{n=1}^{k}\lambda_n^{(m)}\big)\,\mathrm{e}^{-\int_0^{k/T}\psi_m^*(t)\left(\phi_m'(t)\right)\mathrm{d}t}\phi_m(1)\psi_m^*(0) - F^Q(\tfrac{k}{T})\dots F^Q(\tfrac{1}{T})Q(0)\big\| \\ \leq \frac{C}{T(1-\ell')}. \tag{17}$$

In addition,

$$\|F^Q(\tfrac{k}{T})\dots F^Q(\tfrac{1}{T})Q(0)\| \leq C{\ell'}^k$$

where C *depends only on*

$$c_P = \sup_{s\in[0,1]}\max_{m=1,\dots,z}\max\big(\|\phi_m(s)\|, \|\phi_m'(s)\|, \|\psi_m^*(s)\|, \|{\psi_m^*}'(s)\|\big)$$

in a continuous manner, and T_0 *depends only on* c_P *and* ℓ'*.*

Let us give a short sketch of the proof of Theorem 4.6. If one writes $F(s)=F^P(s)+F^Q(s)$, then a product $F(\frac{T}{T})\dots F(\frac{1}{T})$ can be expanded as a sum of products of F^P and of F^Q. We begin by deriving bounds for consecutive products of terms of one or the other kind: there exist T_0, $D>1$ and $D'>1$ that depend only on C_P, such that for $T\geq T_0$

$$\|F^Q(\tfrac{a+L}{T})\dots F^Q(\tfrac{a+1}{T})\| \leq D\,(\ell')^L, \qquad \|F^P(\tfrac{b+L}{T})\dots F^P(\tfrac{b+1}{T})\| \leq D'.$$

The first inequality can be proven using the formula for the spectral radius, which implies that for all s and ℓ' there exists L such that $\|F(s)^L\| \leq \ell'$; if one chooses by compactness a L uniform in s, one can use the fact that $F(s)^L \simeq F(\frac{k+L}{T}) \dots F(\frac{k+1}{T})$ and the regularity **(a1)** of F. The second inequality can be proven using the fact that a product $P^{(m')}(\frac{k+1}{T})P^{(m)}(\frac{k}{T})$ has bound smaller that C_P/T if $m \neq m'$, a combinatorial argument and an adiabatic theorem for products of projectors $P^{(m)}(\frac{k}{T})$. On the other hand, the regularity assumed in **(a4)** implies that products $F^P(\frac{k+1}{T})F^Q(\frac{k}{T})$ or $F^Q(\frac{k+1}{T})F^P(\frac{k}{T})$ also have norms bounded by C_P/T. Anothe combinatorial argument on the number of consecutive P and of Q in the expansion of $F(\frac{T}{T}) \dots F(\frac{1}{T})$ allows to control

$$\big\|F(\tfrac{k}{T}) \dots F(\tfrac{1}{T}) - F^P(\tfrac{k}{T}) \dots F^P(\tfrac{1}{T})P(0) - F^Q(\tfrac{k}{T}) \dots F^Q(\tfrac{1}{T})Q(0)\big\|.$$

There remains to compare $F^P(\frac{k}{T}) \dots F^P(\frac{1}{T})P(0)$ to $\sum_{m=1}^{z} \big(\prod_{k=1}^{T} \lambda^{(m)}(\frac{k}{T})\big)\, W(\frac{T}{T})P^{(m)}(0)$, using arguments similar to those of [119].

Remark 4.8. The results of this section apply on any Banach space B, in particular in infinite dimension. The adiabatic Theorem 4.6 should have other applications, for example in Markov chains with a generator that depends on time in an adiabatic way, at least as soon as the properties under consideration can be characterized in spectral terms. We can for example give a proof of the results of [P24] when the generator Π of (3.34) (and therefore, physically, the measurements carried out on the probes $\mathcal{E}_k$, $k = 1, \dots, T$) evolve adiabatically.

3. Peripheral spectrum of deformations of quantum channels

As announced previously, we can show that the peripheral eigenvalues of $\mathcal{L}^{(\alpha)}(s)$ are simple. To apply Corollary 4.7, we will also need good expressions for the associated peripheral spectral projectors. Remark first that the operators $\mathcal{L}(s) = \mathcal{L}^{(0)}(s)$ are completely positive and trace-preserving.

Consider therefore a completely positive and trace-preserving map Φ on $\mathcal{I}_1(\mathcal{H})$ where $\mathcal{H}$ is of finite dimension, and suppose Φ irreducible. We fix a Kraus representation of $\Phi(\rho) = \sum_{i\in I} L_i \rho L_i^*$ of Φ. Then (see annex A), there exists an integer z, a faithful state ρ_{inv} and a unitary u of the form $\sum_{z=0}^{m-1} \theta^m p_m$ where $\theta = \mathrm{e}^{2\mathrm{i}\pi/z}$, such that the peripheral spectrum of Φ is of the form $S_z := \{\theta^m \mid m = 0, \dots, z{-}1\}$, that each of these peripheral eigenvalues is simple, and that the spectral projector of Φ associated with θ^m is $\eta \mapsto \mathrm{tr}(u^{-m}\eta)\, \rho_{\mathrm{inv}} u^m$.

Let now $(\mathrm{L}_i)_{i\in I}$ be a family of positive reals. We define for $\alpha \in \mathbb{R}$ a map $\Phi^{(\alpha)}$ on $\mathcal{I}_1(\mathcal{H})$ by

$$\Phi^{(\alpha)}(\rho) = \sum_i \mathrm{L}_i^{\alpha}\, L_i \rho L_i^*.$$

Then $\Phi^{(\alpha)}$ is completely positive and satisfies the following properties:

Proposition 4.9 ([P22]). *There exist three smooth maps $\alpha \mapsto \lambda^{(\alpha)}, \mathrm{I}^{(\alpha)}, \rho^{(\alpha)}$ from $\mathbb{R}$ to, respectively, $\mathbb{R}_+^*$, the set of definite positive operators, and the set of faitfhul states, such that for all α in $\mathbb{R}$,*

- *the peripheral spectrum of $\Phi^{(\alpha)}$ is $\lambda^{(\alpha)} S_z = \{\lambda^{(\alpha)}\theta^m \mid m = 0, \dots, z-1\}$,*
- *one has the commutation relations $[\mathrm{I}^{(\alpha)}, u] = 0$ and $[\rho^{(\alpha)}, u] = 0$,*

- *one has* $\operatorname{tr}(\rho^{(\alpha)}\,\mathrm{I}^{(\alpha)}) = 1$ *for all* $\alpha \in \mathbb{R}$,
- *the unique (up to a multiplicative factor) eigenvector of* $\Phi^{(\alpha)}$ *(respectively of* $\Phi^{(\alpha)*}$*) associated with the eigenvalue* $\lambda^{(\alpha)}\theta^m$ *is* $\rho^{(\alpha)}u^m$ *(respectively* $\mathrm{I}^{(\alpha)}u^{-m}$*), and the spectral projector of* $\Phi^{(\alpha)}$ *associated with* $\lambda^{(\alpha)}\theta^m$ *is*
$$\eta \mapsto \operatorname{tr}(\mathrm{I}^{(\alpha)}u^{-m}\eta)\,\rho^{(\alpha)}u^m.$$

Remark 4.10. For $\alpha = 0$ one has $\lambda^{(\alpha)} = 1$, $\mathrm{I}^{(\alpha)} = \mathrm{Id}$ and $\rho^{(\alpha)} = \rho_{\mathrm{inv}}$. Remark also that the maps $\lambda^{(\alpha)}$, $\mathrm{I}^{(\alpha)}$, $\rho^{(\alpha)}$ depend on $(\mathrm{L}_i)_{i\in I}$.

The proof of Proposition 4.9 is long but is essentially based first on the the fact that one knows a bijection between the eigenvectors of $\Phi^{(\alpha)}$ and those of a trace-preserving deformation $\widehat{\Phi}^{(\alpha)}$ of $\Phi^{(\alpha)}$ (this deformation can be found in [56] already); second, that the peripheral spectrum of a trace-preserving map, and some of the properties of the associated spectral projectors, are characterized by commutation relations between the Kraus operators and the spectral projectors of the associated unitary u. One then uses the fact that the Kraus operators of $\Phi^{(\alpha)}$ are proportional to those of Φ to show that these commutation relations remain true, with the same projectors.

Remark 4.11. It seems that Proposition 4.9 was not known before us (see however [69]), maybe because the interest for deformations of quantum channels is a recent matter. We saw that these deformations are of interest as soon as one studies the generating functions associated with indirect measurements. The simple fact that the structure of the peripheral spectrum is preserved simplifies some of the perturbative arguments: it ensures that the property of being an eigenvalue of maximum modulus is preserved under the deformation, and this offers some continuity in our perturbative arguments. This proposition can for example simplify the proof given in [P15] of the results of Theorem 3.16. It is used in the same way in [27].

This characterization through commutation relations also allows to prove that the form of the peripheral spectrum remains stable under perturbations with α complex, with imaginary part small enough that no non-peripheral eigenvalue is deformed into a peripheral eigenvalue. We then have:

Corollary 4.12 ([P22]). *Under the assumptions of Proposition 4.9, there exists for all* α_0 *of* $\mathbb{R}$*, a complex neighbourhood* N_{α_0} *of* α_0*, such that for all* α *in* N_{α_0}*, the peripheral spectrum of* $\Phi^{(\alpha)}$ *is of the form* $\{\lambda^{(\alpha)}\theta^m \,|\, m = 0, \ldots, z-1\}$ *for a* $\lambda^{(\alpha)} \in \mathbb{C}$.

4. Applications: entropy production and Landauer's principle for trajectories of repeated interactions systems

We now return to the study of Landauer's principle, exploiting Proposition 4.5. We will from now on make the assumption that the parameters $\beta(s)$, $H_{\mathcal{E}}(s)$ and $v(s)$ are such that $\mathcal{L}(s)$ is irreducible for all $s \in [0,1]$. Then, if one assumes ρ^{i} faithful, $\rho^{\mathrm{f}} = \mathcal{L}(\frac{T}{T}) \ldots \mathcal{L}(\frac{1}{T})(\rho^{\mathrm{i}})$ is faithful as well, as we needed to assume to obtain (10). We also assume that the M_k are of the form $M_k = M(\frac{k}{T})$ where $s \mapsto M(s)$ is twice continuously differentiable on $[0,1]$ and for all s, $M(s)$ is a function of $\xi(s)$ (or equivalently, of $H_{\mathcal{E}}(s)$). We will denote $\lambda_M(\alpha)$, $\mathrm{I}_M(\alpha)$, $\rho_M(\alpha)$ the functions obtained by an application of Proposition 4.9, to emphasize their dependency in M, and we define $\tilde{\mathcal{L}}_M^{(\alpha)} = (\lambda_M^{(\alpha)})^{-1}\,\mathcal{L}_M^{(\alpha)}$ which therefore has spectral radius equal to 1.

Proposition 4.13 ([P22]). *Consider an* **ARIS** *such that $\mathcal{L}(s)$ is irreducible for all $s \in [0,1]$, with $z(s) \equiv z$. Then there exist continuous functions ℓ', C, T_0 from $\mathbb{R}$ to $(0,1)$, $\mathbb{R}_+$ and $\mathbb{N}$ respectively, T_0 bounded on any compact subset of $\mathbb{R}$, such that for $\alpha \in \mathbb{R}$, $T \geq T_0(\alpha)$, and $k \leq T$,*

$$\begin{aligned} \Big\| \tilde{\mathcal{L}}_M^{(\alpha)}(\tfrac{k}{T}) \dots \tilde{\mathcal{L}}_M^{(\alpha)}(\tfrac{1}{T})\rho^{\mathrm{i}} - z\,\mathrm{e}^{-\vartheta_M^{(\alpha)}} \sum_{m=0}^{z-1} \operatorname{tr}\big(\mathrm{I}^{(\alpha)}(0) p_m(0)\rho^{\mathrm{i}}\big)\, \rho_M^{(\alpha)}(\tfrac{k}{T}) p_{m-k}(\tfrac{k}{T}) \Big\| \\ \leq \frac{C(\alpha)}{T(1-\ell'(\alpha))} + C(\alpha)\ell'(\alpha)^k \end{aligned} \tag{18}$$

where the subscript $m-k$ of the spectral projector $p_{m-k}(\frac{k}{T})$ of $u(\frac{k}{T})$ must be understood modulo z, and

$$\vartheta_M^{(\alpha)} := \int_0^{k/T} \operatorname{tr}\big(\mathrm{I}_M^{(\alpha)}(s)\, \frac{\partial}{\partial s}\rho_M^{(\alpha)}(s)\big)\mathrm{d}s.$$

This proposition can be obtained by combining Corollary 4.7 and Proposition 4.9. We then prove that the integral in the exponential of (17) does not depend on m. The final expression can be derived by a discrete Fourier transform.

We can now consider $\frac{1}{T}\log \Gamma_{(\Delta m_T^{\mathrm{tot}}, \Delta a_T)}$. One has:

$$\log \Gamma_{(\Delta m_T^{\mathrm{tot}}, \Delta a_T)}(\alpha_1,\alpha_2) = \log \sum_{k=1}^{T} \lambda^{(\alpha_1)}(\tfrac{k}{T}) + \log \operatorname{tr}\big((\rho_T^{\mathrm{f}})^{\alpha_1} \tilde{\mathcal{L}}_M^{(\alpha_1)}(\tfrac{T}{T}) \dots \tilde{\mathcal{L}}_M^{(\alpha_1)}(\tfrac{1}{T})\big((\rho^{\mathrm{i}})^{1-\alpha_1}\big)\big).$$

We then use (18) and the expression of $\lim_{T\to\infty} \rho_T^{\mathrm{f}}$ to show that

$$0 < \liminf_{T\to\infty} \operatorname{tr}\big(\tilde{\mathcal{L}}_M^{(\alpha)}(\tfrac{T}{T}) \dots \tilde{\mathcal{L}}_M^{(\alpha)}(\tfrac{1}{T})\rho^{\mathrm{i}}\big) \leq \limsup_{T\to\infty} \operatorname{tr}\big(\tilde{\mathcal{L}}_M^{(\alpha)}(\tfrac{T}{T}) \dots \tilde{\mathcal{L}}_M^{(\alpha)}(\tfrac{1}{T})\rho^{\mathrm{i}}\big) < \infty,$$

which implies immediately that

$$\lim_{T\to\infty} \frac{1}{T} \log \Gamma_{(\Delta m_T^{\mathrm{tot}}, \Delta a_T)}(\alpha_1,\alpha_2) = \int_0^1 \log \lambda^{(\alpha_1)}(s)\,\mathrm{d}s. \tag{19}$$

Remark 4.14. We recover an expression similar to the one in example 3.36 in the time-independent case.

We then obtain:

Theorem 4.15 ([P22]). *Consider an* **ARIS** *such that $\mathcal{L}(s)$ is irreducible for all $s \in [0,1]$, with $z(s) \equiv z$ and assume that the initial state ρ^{i} is faithful. Let*

$$\Lambda_M(\alpha) = \int_0^1 \log \lambda^{(\alpha)}(s)\,\mathrm{d}s.$$

Then $\frac{1}{T}\Delta m_T^{\mathrm{tot}}$ converges exponentially to

$$\Lambda_M'(0) = \sum_{i,j} \int_0^1 \big(y_j(s) - y_i(s)\big)\, \operatorname{tr}\big(K_{i,j}(s)\rho_{\mathrm{inv}}(s)K_{i,j}^*(s)\big)\,\mathrm{d}s,$$

and one has the convergence in distribution

$$\frac{1}{\sqrt{T}}\big(\Delta m_T^{\mathrm{tot}} - T\Lambda_M'(0)\big) \underset{T\to\infty}{\to} \mathcal{N}\big(0, \Lambda_M''(0)\big)$$

(where one has an implicit expression for Λ''_Y), and $\Delta m_T^{\mathrm{tot}}$ satisfies a large deviations principle with rate function Λ_M^* obtained as a Fenchel–Legendre transform of Λ_M:

$$\Lambda_M^*(x) = \sup_{\alpha\in\mathbb{R}} \big(\alpha x - \Lambda_M(\alpha)\big).$$

In all of the above statements, one can replace $\Delta m_T^{\mathrm{tot}}$ by $\Delta m_T^{\mathrm{tot}} + \gamma \Delta a_T$ for any $\gamma \in \mathbb{R}$.

Once again, the large deviations principle follows immediately from the Gärtner–Ellis Theorem using (19), the exponential convergence being a standard consequence. The central limit theorem is obtained using Bryc's Theorem, Corollaries 4.7 and 4.12 allowing us to give a uniform bound of the quantity $\frac{1}{T}\log\Gamma_{(\Delta m_T^{\mathrm{tot}},\Delta a_T)}(\alpha_1,\alpha_2)$ for α_1,α_2 in a complex neighbourhood of the origin.

In particular, when one chooses $M(s) = \beta(s)H_{\mathcal{E}}(s)$, then Theorem 4.15 is a statement about the variable ς_T. We will say that the **ARIS** is time-reversal invariant if for all $s\in[0,1]$ there exist two antiunitary involutions $\vartheta_{\mathcal{S}}(s) : \mathcal{H}_{\mathcal{S}} \to \mathcal{H}_{\mathcal{S}}$ and $\vartheta_{\mathcal{E}}(s) : \mathcal{H}_{\mathcal{E}} \to \mathcal{H}_{\mathcal{E}}$ such that $\vartheta(s) = \vartheta_{\mathcal{S}}(s)\otimes\vartheta_{\mathcal{E}}(s)$ satisfies for all $s\in[0,1]$

$$[H_{\mathcal{S}},\vartheta_{\mathcal{S}}(s)] = 0, \qquad [H_{\mathcal{E}}(s),\vartheta_{\mathcal{E}}(s)] = 0, \qquad [v(s),\vartheta(s)] = 0. \tag{20}$$

Corollary 4.16 ([P22]). *Consider an **ARIS** such that $\mathcal{L}(s)$ is irreducible for all $s\in[0,1]$, with $z(s)\equiv z$ and assume that the initial state ρ^{i} is faithful. The variable ς_T satisfies the exponential convergence, a central limit theorem and a large deviations principle for which the parameters are given by Theorem 4.15 with $M(s) = \beta(s)H_{\mathcal{E}}(s)$; in particular*

$$\Lambda'_{\beta H_{\mathcal{E}}}(0) = \int_0^1 \beta(s)\,\mathrm{tr}\,\big(X(s)\,\mathrm{Id}\otimes H_{\mathcal{E}}(s)\big)\,\mathrm{d}s,$$

where

$$X(s) := U(s)\big(\rho_{\mathrm{inv}}(s)\otimes\xi(s)\big)U(s)^* - \rho_{\mathrm{inv}}(s)\otimes\xi(s). \tag{21}$$

*If the **ARIS** is time-reversal invariant, then the rate function $\Lambda_{\beta H_{\mathcal{E}}}$ satisfies*

$$\Lambda_{\beta H_{\mathcal{E}}}(\alpha) = \Lambda_{\beta H_{\mathcal{E}}}(1-\alpha) \quad \textit{for all } \alpha\in\mathbb{R}.$$

If one has $X\equiv 0$ (i.e. $X(s) = 0$ for all $s\in[0,1]$) then $\frac{1}{T}\varsigma_T$ and $\Delta S_{\mathcal{E}}$ converge exponentially fast to 0; if, on the contrary, $X \not\equiv 0$ they converge exponentially to a strictly positive constant.

Remark 4.17. In particular, if $X\not\equiv 0$, σ_T^{tot} increases linearly in T to $+\infty$. This is the first half of the main result of [P19]. In the case where the system was time-independent and time-reversal invariant, this followed from Theorem 3.27 and from the discussion in example 3.36.

We will continue assuming that $X\equiv 0$ since it is only in that case that one can have saturation; we then try to explicit the relation between the limiting distributions of $\Delta s_{\mathcal{E},T}$ and $\Delta s_{\mathcal{S},T}$. We begin by remarking that for a given $s\in[0,1]$, $X(s) = 0$ if and only if there exists a observable $k_{\mathcal{S}}(s)$ of $\mathcal{H}_{\mathcal{S}}$ such that $[U(s), k_{\mathcal{S}}(s) + H_{\mathcal{E}}(s)] = 0$ (this is Lemma 6.5 of [P19]), in which case one can check that

$$\rho^{(\alpha)}(s) = \frac{\mathrm{e}^{-(1+\alpha)\beta(s)k_{\mathcal{S}}(s)}}{\mathrm{tr}\,\big(\mathrm{e}^{-(1+\alpha)\beta(s)k_{\mathcal{S}}(s)}\big)} \qquad \mathrm{I}^{(\alpha)}(s) = \frac{\mathrm{tr}\,\big(\mathrm{e}^{-(1+\alpha)\beta(s)k_{\mathcal{S}}(s)}\big)}{\mathrm{tr}\,\big(\mathrm{e}^{-\beta(s)k_{\mathcal{S}}(s)}\big)}\,\mathrm{e}^{\alpha\beta(s)k_{\mathcal{S}}(s)}$$

(it is enough to check that each of these terms is invariant by $\mathcal{L}^{(\alpha)}$, $\mathcal{L}^{(\alpha)*}$ respectively, and that the normalization condition $\mathrm{tr}(\rho^{(\alpha)}\mathrm{I}^{(\alpha)}) = 1$) holds. One then obtains $\lambda^{(\alpha)}(s) = 1$. As a consequence, if

$X \equiv 0$, then the mean and the variance in Corollary 4.16 are null, and the rate function $\Lambda^*_{\beta H_{\mathcal{E}}}$ is equal to 0 at 0 and $+\infty$ elsewhere. On the other hand, one can in this case study the convergence in distribution of the pair $\Delta s_{\mathcal{E}}, \Delta s_{\mathcal{S}}$ by returning to the study of the generating function, if we assume that the $\mathcal{L}(s)$ are primitive (as otherwise one can observe oscillations due to the periods). One can prove:

THEOREM 4.18 ([P22]). *Consider an* **ARIS** *such that for all $s \in [0,1]$, $\mathcal{L}(s)$ is irreducible with $z(s) = 1$, $X(s) = 0$, and that the initial state ρ^{i} is faithful. Then the distribution of the pair $(\Delta s_{\mathcal{S},T}, \Delta s_{\mathcal{E},T})$ converges narrowly to the probability measure with generating function*

$$M_{(\Delta s_{\mathcal{E}}, \Delta s_{\mathcal{S}})}(\alpha_1, \alpha_2) = \mathrm{tr}\,\big(\rho_{\mathrm{inv}}(0)^{-\alpha_1} (\rho^{\mathrm{i}})^{1-\alpha_2}\big)\ \mathrm{tr}\,\big(\rho_{\mathrm{inv}}(1)^{1+\alpha_1+\alpha_2}\big),$$

which has finite support. In particular, ς_T converges in distribution to a probability measure with cumulant generating function

$$\log M_{\varsigma}(\alpha) = S_{-\alpha}(\rho_{\mathrm{inv}}(0) \,|\, \rho^{\mathrm{i}}).$$

If in addition $\rho^{\mathrm{i}} = \rho_{\mathrm{inv}}(0)$ then the limiting distribution ς_T is a Dirac measure at zero, so the limiting distribution of the pair $(\Delta s_{\mathcal{E},T}, \Delta s_{\mathcal{S},T})$ is supported by the diagonal.

This result says essentially that in the adiabatic limit one has almost-sure equality of $\Delta s_{\mathcal{S}}$ and of $\Delta s_{\mathcal{E}}$. A similar result in the Hamiltonian framework has been shown by [19].

Remark 4.19. This shows in particular that $\sigma_T^{\mathrm{tot}} \underset{t\to\infty}{\to} S(\rho_{\mathrm{inv}}(0) \,|\, \rho^{\mathrm{i}})$ if $X \equiv 0$. The fact that σ_T^{tot} converged to a finite limit was the other half of the main result of [P19], but at the time we could not identify the limit.

Remark 4.20. If $X \equiv 0$ and the **ARIS** is time-reversal invariant, then the system verifies a detailed balance relation: the dual $\tilde{\mathcal{L}}$ of $\mathcal{L}$ for the scalar product

$$(\eta_1, \eta_2) \mapsto \mathrm{tr}\,\big(\rho_{\mathrm{inv}}^{1/2} \eta_1^* \rho_{\mathrm{inv}}^{1/2} \eta_2\big)$$

verifies, in the notation of (20),

$$\tilde{\mathcal{L}}(\vartheta_{\mathcal{S}} \eta \vartheta_{\mathcal{S}}) = \vartheta_{\mathcal{S}} \mathcal{L}(\eta) \vartheta_{\mathcal{S}},$$

which is one of the usual definitions of the quantum detailed balance (see [P26] and section 2). The condition $X \equiv 0$, however, is stronger that detailed balance, since it does not only involve the reduced dynamics $\mathcal{L}$.

Research projects

My main research projects are related with quantum trajectories. A first natural question would be to study the behaviour of random states $(\rho_n)_n$ in the case where the instruments are not perfect: in that case, the transitions of the Markov chain do not preserve the purity of states, and the sequence $(\rho_n)_n$ has no reason to be asymptotically pure. This question seems to be a difficult one, and we think that the most interesting questions actually stem from the case where the instruments are perfect, which is the case when one makes two-time measurements on the probes. In that case, the study of the sequence $(\rho_n)_n$ reduces to that of the random pure states $(\pi_{\hat{x}_n})_n$. There are then different questions of interest. Even for a fixed choice of measure μ satisfying relations (32) and (33) (for example if one fixes the measured probe observable M, in which case μ is fixed with finite support), there remains a lot to say. First, regarding the behaviour of the sequence $(\hat{x}_n)_n$ beyond convergence in distribution: the results of [P24] already allow us to prove a "law of large numbers" improving the Kümmerer–Maassen ergodic Theorem by taking into accountaverages as for example $\frac{1}{n}\sum_{k=1}^{n} f(v_k, \hat{x}_k)$, as well as a functional central limit theorem. It would be useful to obtain a large deviations principle for $(\hat{x}_n)_n$ but this is at the moment out of reach, as the information we have on the spectrum of the transition kernel insufficient: one can hope to identify sub-additivity properties to use a Ruelle–Lanford approach. In the same way, we don't know anything about the properties of the invariant measure ν_{inv} yet; it would be interesting in particular to know how it behaves, and if there is a concentration of measure phenomenon when, for example, the dimension of the system or that of the probes, grows, and depending on whether these probes model an environment at equilibrium or out of equilibrium. One could expect a connection between the properties of $(\hat{x}_n)_n$ or of ν_{inv}, and those of the two-time measurements statistics. One could also study non-Markovian versions of quantum trajectories. The long term interest of these questions is to study and to characterize the evolution of quantum systems through the sequences $(\hat{x}_n)_n$, which could play the role of trajectories in phase space. The bet here is that the decomposition $\int |x\rangle\langle x| \,\mathrm{d}\nu(\hat{x})$ of a state ρ can make physical sense thanks to indirect measurements; for a fixed interaction between system and probe, the dependence of these results in the choice of the measured observables can be taken into account by choosing the latter randomly, this choice being incorporated into the measure μ.

My other research projects concern questions which have not been described here. First, with motivations close to the topics discussed in this manuscript, is the investigation of the properties of transport, or of propagation of intrication and information in other models of quantum systems. A first interesting model is that of "random quantum circuits" which are extremely simple to describe where the randomness allows to consider "typical" evolutions on a quantum spin chain. Another interesting model is that of spin chains to which one adds a strong noise, that seem to display universality properties. Last, other problems, which lie further from the topics discussed in this manuscript, appeared as technical questions in my past work and seem to raise intesting mathematical questions. I am thinking, first, of attempts to generalize the hypothesis testing results from [P16] to general von Neumann algebras, that seem to require decompositions of type QR on such algebras. I am also thinking of more recent work on quantum generalizations of Wasserstein distances, viewed from the angle of couplings of quantum states, and which even in the simplest cases, raise surprisingly difficult questions in relation with the problem of quantum marginals, which is a standard topic of quantum information theory.

Appendix A

Completely positive maps and their peripheral spectrum

Let us begin by recalling that for $\mathcal{H}$ a separable Hilbert space, a map Φ from $\mathcal{I}_1(\mathcal{H})$ to itself is:

- *completely positive* if for all $n \in \mathbb{N}^*$, its extension $\Phi \otimes \mathrm{Id}_{\mathcal{B}(\mathbb{C}^n)}$ to $\mathcal{I}_1(\mathcal{H}) \otimes \mathcal{B}(\mathbb{C}^n)$ is positive; the complete positivity therefore implies positivity.
- *irreducible* if the only orthogonal projectors P of $\mathcal{H}$ satisfying $\Phi\big(P\mathcal{I}_1(\mathcal{H})P\big) \subseteq P\mathcal{I}_1(\mathcal{H})P$ are 0 and $\mathrm{Id}_{\mathcal{H}}$,

For a motivation to consider completely positive maps, see [85]. A map that is both completely positive and trace-preserving (i.e. such that $\mathrm{tr}\,\big(\Phi(\eta)\big) = \mathrm{tr}(\eta)$) is often called a *quantum channel*. We recall that any completely positive map admits a *Kraus decomposition*, that is, there exist operators $L_i \in \mathcal{B}(\mathcal{H})$, where i belongs to an at most countable family I, such that $\Phi(\rho) = \sum_{i\in I} L_i \rho L_i^*$ for all ρ (see [85] or [110] for a more modern presentation). The irreducibility of Φ is then equivalent to the fact that the only vector subspaces E of $\mathcal{H}$ such that $L_i E \subset E$ for all $i \in I$ are $\{0\}$ and $\mathcal{H}$.

Suppose that Φ is completely positive and irreducible. A first result holds in any dimension (Theorem 3.1 of [117]): if the spectral radius $\lambda(\Phi)$ of Φ is equal to its operator norm $\|\Phi\|$ and is an eigenvalue of Φ, then it is a simple eigenvalue, and an eigenvector is definite positive. One has $\|\Phi^n\| = \sup \mathrm{tr}\,\big(\Phi^n(X)\big)$, where the supremum is taken over the set of definite positive X of trace 1, so if Φ is trace-preserving, then $\|\Phi\| = 1$ and $\lambda(\Phi) = \lim_{n\to\infty} \|\Phi^n\|^{1/n} = 1$; one has $\lambda(\Phi) = \|\Phi\| = 1$. If $\mathcal{H}$ is of finite dimension, then the condition $\lambda(\Phi) = \|\Phi\|$ is automatically true.

From now on we suppose that $\mathcal{H}$ is of finite dimension. If Φ is completely positive, irreducible, and trace-preserving (in which case Φ^* is completely positive, irreducible, and preserves Id), then as Evans and Hoegh-Krøhn showed in [56]:

- the peripheral spectrum of Φ is a finite subgroup $S_z = \{\theta^m \,|\, m = 0, \ldots, z-1\}$ (where $\theta = \mathrm{e}^{2\mathrm{i}\pi/z}$) of the unit circle, and each θ^m is a simple eigenvalue,
- there exists a faitful state ρ_{inv}, and a unitary u (called Perron–Frobenius unitary of Φ) satisfying $[\rho_{\mathrm{inv}}, u] = 0$, $u^z = \mathrm{Id}$ and $u^k \neq \mathrm{Id}$ for $k = 0, \ldots, z-1$, such that

$$\begin{aligned} \Phi(\rho u) &= \theta\, \Phi(\rho) u \quad \forall \rho \in \mathcal{I}_1(\mathcal{H}) \\ \Phi^*(uX) &= \bar{\theta} u\, \Phi^*(X) \quad \forall X \in \mathcal{B}(\mathcal{H}). \end{aligned} \tag{1}$$

An immediate consequence is that the unique (up to a multiplicative constant) eigenvector of Φ (respectively Φ^*) associated with the eigenvalue θ^m is $\rho_{\mathrm{inv}} u^m$ (respectively u^{-m}), and the spectral projector of Φ associated with θ^m is $\eta \mapsto \mathrm{tr}(u^{-m}\eta)\rho_{\mathrm{inv}} u^m$.

Remark that the relations (1) are equivalent to $L_i u = \theta u L_i$ for all $i \in I$. In addition, u can be written $u = \sum_{m=0}^{z-1} \theta^m p_m$, where the projectors p_m satisfy $p_m L_i = L_i p_{m+1}$ for all i and m ($m+1$ being understood *modulo* z). Each subspace $\mathcal{B}(\mathrm{Im}\, p_m)$ of $\mathcal{B}(\mathcal{H})$ is therefore invariant by Φ^{*z} and the restriction of Φ^{*z} to that subspace is *primitive*, i.e. irreducible with 1 as a unique peripheral eigenvalue.