Mini-cours de l’IHES

The Frobenius Structure Conjecture for Log Calabi-Yau Varieties 3/4

Tony Yue YU

Université Paris-Sud
Ref: arXiv 1908.09861 joint w S. Keel

Plan: 1. Deformation invariance of naive counts
2. Walls, spines, tropical curves, and tropical moduli spaces
3. Sketch of proof of the connected component theorem
4. Toric tail conditions in families
5. Gluing formula and independence on the choice of torus
6. Structure constants and associativity of mirror algebra
7. Convexity and finiteness
8. Boundary torus action and finite generation

1. Deformation invariance of naive counts
Recall from the last lecture:
Given any spine \( h : \Gamma \to \text{Sk}(U) \) and curve class \( \beta \) \( \rightsquigarrow \) naive count of skeletal curves \( N(h, \beta) \) of spine \( h \) and class \( \beta \).

The definition was straightforward once we have the skeletal curve theorem.

**Question:** what properties do the numbers \( N(h, \beta) \) enjoy?

**The most-wanted property:** Deformation invariance:

We want the invariance of the count \( N(h, \beta) \) under small deformations of \( h \).

**Remark:** This property determines directly the viability of the whole project of non-archimedean curve counting. So it was the first property that we had to check before embarking on the project.

**Rough idea:** Consider \( \{ \text{analytic curves in } U^n \} \)

\[
\begin{array}{ccc}
\text{Sp} & \text{taking associated spine} \\
\Downarrow \\
h \in & \{ \text{spines in } \text{Sk}(U) \}
\end{array}
\]

Intuitively, the count \( N(h, \beta) \) is the cardinality of the fiber \( \text{Sp}^{-1}(h) \) (We ignore curve classes here.)

So for the invariance of \( N(h, \beta) \) under small deformation of \( h \), we want the map \( \text{Sp} \) to be somewhat “étale” over a neighborhood of \( h \).

**More precisely:** Recall that the counts \( N(h, \beta) \) are defined via evaluation of an internal marked point \( p_i \).

Assume for simplicity that \( h \) is an extended spine.

Let \( P := (P_j)_{j \in J} \) be the weight vectors at \( \infty \).

Recall \( P_j \in \text{Sk}(U, \mathbb{Z}) = \{ 0 \} \cup \{ m \nu : m \in \mathbb{N}_+, \nu \text{ essential divisors valuation} \} \).

Recall \( V_1 \) and \( V_2 \) have disjoint closures and the projection to the skeleton is a local isomorphism for all \( U \).
If $P_j = 0$, $j$ is called *internal*.

If $P_j \neq 0$, $j$ is called *boundary*, we write $P_j = m_j \nu_j$ and assume that each $\nu_j$ is given by a component $D_j \subset D$.

We have a *proper* moduli stack:

$$\overline{M}(Y, P, \beta) := \left\{ \text{rational stable maps } f : [C, (P_j)_{j \in J}] \to Y \text{ of class } \beta, \right. $$

\[\text{s.t. each boundary marked point } p_j \text{ maps to } D_j \text{ with order } \geq m_j. \]

Inside $\overline{M}(Y, P, \beta)$ we have the substack $M(U, P, \beta) \subset \overline{M}(Y, P, \beta)$ more relevant to our counts, consisting of curves whose intersection with $D$ is exactly given by $P$, i.e. each boundary marked point $p_j$ maps to $D_j^0$ with order $= m_j$, and no other intersections with $D$.

For any internal marked point $p_i$ (i.e. $p_i \to U$) we have the natural map

$$\Phi_i := (st, ev_i) : \overline{M}(Y, P, \beta) \to \overline{M}_{g,n} \times Y$$

*stabilization of domain.*

In order to obtain the *étaleness* of $\Phi_i$, we consider two more substacks:

$M^{st}(U, P, \beta) \subset M(U, P, \beta)$ consisting of stable maps with stable domain (i.e. no bubbles)

$M^{sm}(U, P, \beta) \subset M^{st}(U, P, \beta)$ consisting of stable maps s.t. $f^* T_Y (-\log D)$ is trivial

These two substacks are in fact spaces (i.e. non-stacky) because stable pointed rational curves do not have nontrivial automorphisms.

Using the deformation theory of curves, we have the following:

**Theorem (smoothness):**

1. $\Phi_i$ is étale over $M^{sm}(U, P, \beta)$

2. $M^{sm}(U, P, \beta)$ is sufficiently big:

For any $\mu \in \overline{M}_{g,n}$, there is a Zar dense open $V \subset U$ s.t.

$$M(U, P, \beta)_\mu, V \subset M^{sm}(U, P, \beta)_\mu.$$
Now consider the following commutative diagram:

\[
\begin{array}{rcl}
\overline{M}^\text{sm}(U^\text{an}, P, \beta) & \xrightarrow{\Phi_i} & \overline{M}_0^\text{an} \times U^\text{an} \\
\downarrow \text{Sp} & & \downarrow \text{P} \\
\text{SP}(M_{\text{IR}}, P) & \xrightarrow{\Phi_i^{\text{trop}}} & \overline{M}_0^{\text{trop}} \times \overline{M}_{\text{IR}}
\end{array}
\]

the space of spines in $\text{Sk}(U) \cong M_{\text{IR}}$ with infinite directions $P$

\[\text{recall } U \supset T_M \text{ torus with cocharacter lattice } M\]

**Theorem (connected component):** Let $S \in \text{SP}(M_{\text{IR}}, P)$ be a transverse extended spine. There exists an open connected neighborhood $V_S$ of $S$ and a Zariski dense open $R \subset \text{P}^{-1}(\Phi_i^{\text{trop}}(V_S))$ s.t. $M_{V_S, R} := M^\text{sm}(U^\text{an}, P, \beta)_{V_S, R}$ is a union of connected components of $\overline{M}(Y^\text{an}, P, \beta)_R$.

**Immediate consequence:** By the properness of $\overline{M}(Y^\text{an}, P, \beta)$ and the smoothness theorem, we deduce that: The restriction $M_{V_S, R} \xrightarrow{\Phi_i} R$ is finite étale, whose degree = the count $N_i(S, \beta)$.

Hence $N_i(S, \beta)$ is constant for all $S \in V_S$.

**Remark:** 1) This shows the invariance of the count $N(S, \beta)$ under small deformation for a transverse extended spine $S$. Then we prove for all transverse spines by studying the toric tail condition in families (to be explained later in the lecture). Deformation invariance does not hold in general for non-transverse spines, even though for the proofs of associativity and wall-crossing homomorphism, we will need a slight generalization for “almost” transverse spines.
2) Actually, we had to prove a stronger version of the connected component theorem: we can furthermore assume that \( R \) intersects every fiber of the projection

\[
\rho^{-1}(\Phi_i^\text{top}(V_S)) \rightarrow \overline{M}_{g,n},
\]

in other words, when restricting to \( R \), we are not going to miss any moduli of the domain curve. This will be important for the proof of the gluing formula, where we need to consider degenerate domain moduli like \( \infty \), which à priori may be missed when we restrict to a Zariski open \( R \).

3) Recall: Finite étale \( \iff \) proper + étale

Properness is a question of compactification while étaleness is a question of transversality. Compactification and transversality are two pivotal themes of enumerative geometry. Ideally, we would like to treat them separately. However, here the proofs of the two properties are intertwined for two reasons:

(i) If we want the stronger version in Remark 2) (i.e. not to miss any domain modulus), we must apply the smoothness theorem above with some properness conditions.

(ii) Properness prevents analytic curves in \( U \) from escaping to \( \infty \) (i.e. to \( D \subset Y \)). If we want to establish properness purely via tropicalization, we must show that tropical curves in \( \text{Sk}(U) \) do not escape to \( \infty \) (i.e. to \( \partial \text{Sk}(U) = \text{Sk}(U) \setminus \text{Sk}(U) \)), but it is extremely complicated to consider tropical curves with components in \( \partial \text{Sk}(U) \), especially the moduli space of such curves. The task is greatly simplified if we use the smoothness theorem within the proof of properness.
2. Walls, spines, tropical curves and tropical moduli spaces

In order to better understand the behavior of non-archimedean analytic curves, we need to study the associated tropical curves.

**Question:** How do we take tropicalization of analytic curves in $U^\text{an}$?

Recall that skeletal curves have canonical spines. What about non-skeletal curves?

Moreover, the spine is only part of a bigger tropical curve, so how do we obtain the whole tropical curve?

**Idea:** Unlike spines of skeletal curves, tropicalizations are not canonical in general, they depend on the choice of some “model”. Here we will work with “toric models”.

Recall: $T_M \subset U \subset Y, \quad D = Y \setminus U$.

**Lemma (toric model):** (After replacing $(Y, D)$ by some toric blowup)

$\exists$ toric compactification $(Y_t, D_t)$ of $T_M$ st. the birational map $\pi: Y \dashrightarrow Y_t$ induces a bijection between the generic points of the strata of $D$ and the generic points of the strata of $D_t$.

Intuitively, $(Y, D)$ is simply a blowup of some toric $(Y_t, D_t)$ whose center does not contain any strata of $D_t$.

**Example:**

![Toric Fan](attachment:toric_fan.png) $\quad$ ![Blowup](attachment:blowup.png) $\quad$ ![Compactification](attachment:compactification.png)

**Notation:** Let $E \subset Y$ and $E_t \subset Y_t$ denote the complement of the isomorphism loci of $\pi$.

In the above example, $E = \begin{array}{c} - \\ \end{array}$, $E_t = \begin{array}{c} - \\ \end{array}$. 
Recall of tropicalization of curves in the toric case:

Given \((Y_t, D_t)\) toric with structure torus \(T_M\) \(M: \) cocharacter lattice, e.g. \(T_M = M \otimes \mathbb{C}^n\)

We have valuation map \(T_M^{\text{an}} \rightarrow M_{\mathbb{R}} \approx \mathbb{R}^n\) compactifies to \(\overline{Y_t}^{\text{an}} \rightarrow \overline{M}_{\mathbb{R}}\).

Compactification of \(M_{\mathbb{R}}\) according to the fan of \((Y_t, D_t)\).

Now given any analytic curve \(C \xrightarrow{\mathcal{f}} Y_t^{\text{an}},\) recall \(C \sim \infty - \text{graph}\).

We have factorization \(C \xrightarrow{\mathcal{f}} Y_t^{\text{an}} \quad \xrightarrow{\mathcal{E}} \quad \Gamma \xrightarrow{h} \overline{M}_{\mathbb{R}}\)

where \(\mathcal{E}: C \rightarrow \Gamma\) contracts every path in \(C\) that are contracted by \(h \circ \mathcal{f}\).

Fact from tropical geometry:

\(\Gamma\) is a finite metric graph, \(h\) is piecewise affine and balanced—meaning that the sum of weight vectors (i.e. derivative) around every vertex is 0.

So \(h: \Gamma \rightarrow \overline{M}_{\mathbb{R}}\) is essentially a combinatorial object, which is called the tropical curve associated to the analytic curve \(\mathcal{f}: C \rightarrow Y_t^{\text{an}}\).

Rem: If we have marked points \(\{p_j\} \subset C\), let \(\Gamma^5 \subset C\) be the convex hull of \(\{p_j\}\).

We call the restriction \(\Gamma^5 \xrightarrow{\mathcal{E} \circ \mathcal{f}} \overline{M}_{\mathbb{R}}\) the associated spine.

In this case, in the definition of tropical curve, it is natural to require that the contraction \(\mathcal{E}: C \rightarrow \Gamma\) does not contract any edge of \(\Gamma^5\), so that \(\Gamma^5\) is a subset of \(\Gamma\). (In other words, \(h\) is an immersion off spine.)

The branches of \(\Gamma\) off spine are called twigs.

This is how to tropicalize analytic curves in \(Y_t^{\text{an}}\).
In order to tropicalize analytic curves in our original $Y^\text{an}$, we simply compose with the toric model $\pi: Y \rightarrow Y^\text{t}$ (it is possible to deal with the indeterminate locus of $\pi$, but let's ignore this for now.)

**Question:** What do tropicalization of analytic curves in $M(U, P, \beta)$ look like?

**Example:**

$tropicalize$

Observation: Since our analytic curve meets $D$ only at the marked points $\{P_i\}$, the twigs cannot meet $\overline{\Delta M}_R$ at arbitrary places, because most of $\overline{\Delta M}_R$ is just tropicalization of $D$, except the subset $E^\text{top} \subset \overline{\Delta M}_R$ of codim $\geq 1$.

**Definition:** Let $\text{Wall} \subset M^R$ be the image of all possible twigs. Then balancing condition + above observation

$\Rightarrow$ $\text{Wall} \subset M^R$ is polyhedral of codim $\geq 1$

(we can make it finite polyhedral by bounding the degree of twigs.)

**Next observation:** By the balancing conditions again, a spine can only bend at $\text{Wall}$, i.e. a vertex of a spine is balanced unless it lies in $\text{Wall}$. This gives strong constraints on the shape of spines, especially for transverse spines.
Since $\text{Wall} < M_{\mathbb{R}}$ is of codim $\geq 1$, we see that for a transverse spine $h: \Gamma \to M_{\mathbb{R}}$, if we require $h(\Gamma)$ to pass through a fixed point $x \in M_{\mathbb{R}}$, then we can no longer deform the map $h$ (i.e. it becomes rigid). Furthermore, if we perturb a little bit $x \in M_{\mathbb{R}}$, then $h: \Gamma \to M_{\mathbb{R}}$ deforms uniquely. This is called the rigidity property of transverse spines. Let's give the precise statement: \(^{1}\) (for simplicity we restrict to extended spines)\(^{1}\)

\begin{align*}
\text{Rigidity property:} & \quad \text{Consider } \Phi_{i}^{\text{tr}} := (\text{dom}, \text{ev}_{i}): SP^{\text{tr}}(M_{\mathbb{R}}, P) \to M_{\mathbb{R}}^{\text{tr}} \times M_{\mathbb{R}} \\
& \quad \text{internal marked point, i.e. } P_{i} = 0 \\
& \quad \text{space of transverse spines in } M_{\mathbb{R}} \text{ with } \infty \text{ directions } P \\
& \quad \text{Let } S \in SP^{\text{tr}}(M_{\mathbb{R}}, P) \quad \text{Then there exists a connected open neighborhood } S \in V_{S} \subset SP^{\text{tr}}(M_{\mathbb{R}}, P) \text{ such that } \\
& \quad \text{the restriction of } \Phi_{i}^{\text{tr}} \text{ to } V_{S} \text{ is a homeomorphism onto its image, and is open. } \\
\end{align*}

\begin{align*}
\text{Remark:} & \quad \text{Let } \overline{S} \text{ be the image of } S \text{ in } \overline{M}_{0,n}^{\text{tr}} \times M_{\mathbb{R}}, \\
& \quad \text{and } V_{\overline{S}} \text{ the image of } V_{S} \text{ in } \overline{M}_{0,n}^{\text{tr}} \times M_{\mathbb{R}}. \\
\text{Rigidity property } & \Rightarrow V_{S} \text{ is a connected component of } SP^{\text{tr}}(M_{\mathbb{R}}, P)_{V_{\overline{S}}} \quad \text{(1)} \\
& \text{This looks familiar to the connected component theorem above (for the purpose of deformation invariance), which claimed that } \\
& \quad M_{V_{S}, R} := M^{\text{sm}}(U^{an}, P, \beta)_{V_{S}, R} \text{ is a union of connected components of } \overline{M}(U^{an}, P, \beta)_{R}. \quad \text{(2)} \\
& \text{Let's recall the notations: we had commutative diagram:}
\end{align*}
Assume we extend $Sp$ to some $\overline{Sp}: \overline{M}(Y^n, P, \beta) \rightarrow SP(\overline{M}_R, P)$, then by some hypothetical continuity of $\overline{Sp}$, the connected component statement $\overline{1}$ implies immediately that $\overline{M}(Y^n, P, \beta)_{V_S} \subset \overline{M}(Y^n, P, \beta)_{V_{\overline{S}}}$ is a union of connected components. Then statement $\overline{2}$ seems to follow from the smoothness theorem.

Unfortunately, this reasoning has two major flaws:

1) Note statement $\overline{1}$ was only about transverse spines, but we applied it to all spines. Non-transverse spines can be terrible (imagine a spine lies completely in Wall, then the bends and the weights are out of control), consequently, there is no reasonable moduli space of non-transverse spines.

Nonetheless, tropical curves always satisfy the balancing condition, and have a nice moduli space. In fact, we can prove that

$$\overline{TC}(M_R, P)_{V_S} \subset \overline{TC}(M_R, P)_{V_{\overline{S}}}$$

is a union of connected components.

It still falls short for remedying the flawed proof, because we must consider not only tropical curves in $M_R$, but also in $\overline{M}_R$ (i.e. allowing components to go to $\overline{\partial M}_R$). This is not theoretically impossible, but we gave up because the combinatorics become too complicated.
2) Spine alone cannot guarantee that a stable map \( \{f: C \to Y^a\} \in \overline{M}(Y^a, P, \beta) \) lies in \( \overline{M}(U^a, P, \beta) \) (i.e. \( f(C) \) meets \( D^a \) only at the marked points), because entire component of \( C \) may lie inside \( D^a \). So we cannot really apply the smoothness theorem to conclude.

One can try to remedy this flaw by considering tropical curves in \( \overline{M}_{\mathbb{R}} \), but it is not easy.

After all these discussions, we are finally ready to sketch the proof that works!

3. Sketch of proof of the connected component theorem

First we need:

**Theorem (continuity of tropicalization):** The tropicalization map

\[
\text{Trop}: \mathcal{M}^{sm}(U^a, P, \beta) \longrightarrow \text{TC}(\overline{M}_{\mathbb{R}}, P)
\]

is continuous.

The composite map \( \text{Sp}: \mathcal{M}^{sm}(U^a, P, \beta) \longrightarrow \text{TC}(\overline{M}_{\mathbb{R}}, P) \longrightarrow \text{SP}(\overline{M}_{\mathbb{R}}, P) \) is continuous over \( \text{SP}^{tr}(\overline{M}_{\mathbb{R}}, P) \), the locus of transverse spines.

**Remark:** 1) We claim continuity only over the locus of transverse spines because there is no nice topology on the set of non-transverse spines.

2) This continuity statement follows from a general continuity result in Yu 2015 or Ranganathan 2017, which is proved using formal models and log geometry respectively. It is also easy to give a direct proof in this special case.

Now we recall the statement of the connected component theorem:

We have commutative diagram:
Let \( S \in \text{SP}(M_{\mathbb{R}}, \mathcal{P}) \) be a transverse extended spine. Then there exists an open connected neighborhood \( V_S \) of \( S \) and a connected Zariski open \( R \subset \rho^{-1}(\Phi_i^{\text{top}}(V_S)) \) s.t. \( M_{V_S, R} := \mathcal{M}^{\text{sm}}(U^\alpha, \mathcal{P}, \beta)_{V_S, R} \) is a union of connected components of \( \overline{\mathcal{M}}(Y^\alpha, \mathcal{P}, \beta)_R \).

**Sketch of proof:**

**Step 1:** The rigidity property of transverse spine

\[ \Rightarrow \exists \text{ an open connected neighborhood } S \in V_S \subset \text{SP}^{\text{tr}}(M_{\mathbb{R}}, \mathcal{P}) \text{ such that } V_S \subset \text{SP}^{\text{tr}}(M_{\mathbb{R}}, \mathcal{P})_{V_S} \text{ is a union of connected components.} \]

\[ \Rightarrow \text{Up to shrinking } V_S, \quad TC(M_{\mathbb{R}}, \mathcal{P})_{V_S} \subset TC(M_{\mathbb{R}}, \mathcal{P})_{V_S} \text{ is a union of conn. comp.} \]

By the continuity theorem

\[ \Rightarrow M^{\text{sm}}(U^\alpha, \mathcal{P}, \beta)_{V_S} \subset M^{\text{sm}}(U^\alpha, \mathcal{P}, \beta)_{V_S} \text{ is a union of conn. comp.} \]

\[ \Rightarrow M^{\text{sm}}(U^\alpha, \mathcal{P}, \beta)_{V_S} \subset \overline{\mathcal{M}}(Y^\alpha, \mathcal{P}, \beta)_{V_S} \text{ is Zar open.} \]

So it remains to show that it is Zar closed after restricting to some dense open \( R \subset \rho^{-1}V_S \).

**Step 2:** Claim: Let \( \varphi: \Delta \rightarrow \overline{\mathcal{M}}(Y^\alpha, \mathcal{P}, \beta)_{V_S} \) s.t. \( \varphi(\Delta \setminus o) \subset M^{\text{sm}}(U^\alpha, \mathcal{P}, \beta)_{V_S} \), then \( \varphi(o) \in \mathcal{M}(U^\alpha, \mathcal{P}, \beta) \).

**Idea of proof of claim:**
Step 2.1: We show that there exists a compact analytic domain $K \subset U^\infty$ such that the boundary of each cap of every fiber maps to $K$.

Idea: Use the rigidity property of transverse spine + the continuity theorem

Step 2.2: We show that the body of each fiber maps to $K$, in particular, the body is disjoint from $D^\infty \subset Y^\infty$. 
Idea: Use the affineness of $U$ and the maximum modulus principle. We cannot prevent the body from touching the boundary without the affineness of $U$.

Step 2.3: We show that up to shrinking the size $\varepsilon$ of the caps, the caps of the central fiber cannot contain any bubbles.

Idea: Use the fact that $\text{Wall} \subseteq \text{Sk}(U)$ has codim $\geq 1$, the continuity theorem and the stability condition of stable maps.

Steps 2.1-2.3 $\Rightarrow$ the stable map $\varphi(0)$ meets $D^m$ only at the marked points, and the tangency orders are also good by the curve class $\beta$. $\Rightarrow$ the claim in Step 2.

**Step 3:** Conclude from Steps 1-2 and the smoothness theorem.

Recall from Step 1: $M^\text{sm}(U^m, P, \beta)_V \subseteq \overline{M}(Y^m, P, \beta)_V$ is Zariski open.

It remains to show that it is Zar closed after restricting to some dense open $R \subseteq \rho^{-1}V_\delta$.

Recall from Step 2: $\forall \varphi : \Delta \to \overline{M}(Y^m, P, \beta)_V$ with $\varphi(\Delta \setminus 0) \subseteq M^\text{sm}(U^m, P, \beta)_V$, we have $\varphi(0) \in M(U^m, P, \beta)$.

$\Rightarrow$ The Zariski closure of $M^\text{sm}(U^m, P, \beta)_V$ in $\overline{M}(Y^m, P, \beta)_V$ lies in $M(U^m, P, \beta)_V$.

We denote it by $\overline{M}_V$.

We obtain a proper map $\overline{M}_V \xrightarrow{\Phi_i} \rho^{-1}(V_\delta)$ by the properness of $\overline{M}(Y^m, P, \beta)$.

Claim: We can take $R$ to be the complement of $\Phi_i(\overline{M}_V \setminus M^\text{sm}(U^m, P, \beta))$.

Proof: We have $(\Phi_i \mid \overline{M}_V)^{-1}(R) \subseteq M^\text{sm}(U^m, P, \beta)$ by construction.

So the Zariski closure of $M^\text{sm}(U^m, P, \beta)_V$, $R$ in $\overline{M}(Y^m, P, \beta)_R$ lies in $M^\text{sm}(U^m, P, \beta)_R$. So $M^\text{sm}(U^m, P, \beta)_V$, $R$ is a union of connected components of $\overline{M}(Y^m, P, \beta)_R$.  

Note: Smoothness theorem $\Rightarrow R$ intersects every fiber of the projection $p^{-1}(V_5) \to \overline{M}_{0,n}^{an}$. In particular, $R \neq \emptyset$.

This completes the proof of the connected component theorem.

4. Toric tail conditions in families

Recall that for counting curves with boundaries associated to truncated spines, we must impose some extra regularity conditions on the boundary, so as to obtain a finite dimensional moduli space. When our log Calabi-Yau variety $U$ contains an algebraic torus $T_m$, we can take advantage of this torus, and impose a simple boundary regularity condition called toric tail condition. We have introduced the toric tail condition for skeletal curves in the two previous lectures. For the deformation invariance of counts associated to truncated spines, we must study toric tail conditions in families.

First let us extend the toric tail condition to non-skeletal curves.

**Question:** How to specify a tail inside a rational curve?

Recall from the symmetry theorem that adding or removing internal marked points does not affect the counts, so we can specify tails by adding as many extra internal marked points as we want.

**Example:**

Consider a rational analytic curve $C$ with marked points $p_1, \ldots, p_4$. The red subtree is their convex hull. If we want to specify a tail containing some boundary marked point $p_e$, e.g. $e=2$, we add an internal marked point $p_5$, and consider $T := r^{-1}[p_5, p_e]$, where $r$ denotes the retraction map from $C$ to the convex hull of all the marked points. If we want $T$ to be a genuine tail containing $p_e$, we should choose $p_5$ sufficiently close to $p_e$ so that $T$ does not contain other
boundary marked points. We denote by $T^* := T \setminus \{p_e\}$ the punctured tail.

Recall that the toric tail condition asks the punctured tail to map to $T^\text{an} \subset U^\text{an}$.

The example shows that in general if we want to impose tail conditions for stable maps in our moduli space $M^\text{sm}(U^\text{an}, P, \beta)$, we can specify a tail by picking an internal marked point $p_s$ and a boundary marked point $p_e$, and we should restrict to the subsat $\Theta \subset \overline{M}^\text{an}_{0,n}$ where the preimage $T := r^{-1}[p_s, p_e]$ does not contain other boundary marked points.

**Lemma (toric tail condition equivalent formulations):** For $f \in M^\text{sm}(U^\text{an}, P, \beta)_{\Theta}$, TFAE:

1. $f(T^*) \subset T^\text{an}$
2. $f(T) \subset \text{isomorphism locus of the toric model } \pi: (Y, D) \dasharrow (Y_t, D_t)$
3. $f(T) \cap E = \emptyset$, where $E$ is the $\pi$-exceptional locus
4. There are no twigs of the tropical curve $\text{Trop}(f)$ attached to the path $[p_s, p_e] < \text{the spine}$.

**Proposition (toric tail outside Wall):** Let $N \subset M^\text{sm}(U^\text{an}, P, \beta)_{\Theta}$ be a subspace s.t. $f(p_s) \notin \text{Wall}$ for all $f \in N$. Then the subspace satisfying the toric tail condition $N\text{tail} \subset N$ is a union of connected components.

**Proof:** Since the toric tail condition equivalent formulation (2) is an open condition, the openness of the inclusion follows. For closedness we use the equivalent formulation (4) and pick any...
sequence (or more precisely any net) $f_\lambda \in N_{tail}$ converging to some $f \in N$.

For any $f_\lambda$, consider the associated tropical curve:

by toric tail condition (4), there are no twigs attached to the path $[P_s, P_e]$ for any $f_\lambda$, in particular the leg $L_s$ containing $P_s$ must be contracted.

Then by the continuity of tropicalization, for the limit $f$, the leg $L_s$ is also contracted. Moreover, if any twig moves into the path $[P_s, P_e]$ under the limit, the twig must attach to the vertex $P_s := L_e \cap L_s$. This is impossible because $f(P_s) \notin Wall$ and the leg $L_s$ is contracted. So $f$ must satisfy toric tail condition (4), hence the closedness holds.

**Remark:**
1) This proposition + the connected component theorem

$\Rightarrow$ the invariance of the count $N(S, \rho)$ under small deformations of any transverse spine. (Here the spine can be truncated, and “transverse” means in particular that the finite 1-valent vertices do not map to $Wall \subset Sk(u)$)

2) For the proof of the associativity of the mirror algebra and the proof of the wall-crossing homomorphism, we will need deformation invariance for not only transverse spines, but also certain non-transverse spines.

E.g. for the associativity of the mirror algebra, we need to show that the spines $S_e$, $S_m$ and $S_r$ have the same counts: imagine $S_e$ deforms to $S_m$ and then to $S_r$. 
Similarly, for the proof of wall-crossing homomorphism (to be explained in the next lecture), when we need to verify the ring homomorphism if we multiply by a theta function that is “parallel” to a wall, we will need to show some deformation invariance as follows:

\[ \text{Sr} \quad \text{Sm} \quad \text{Se} \]

\text{Sm is non-transverse}

**Observation:** In both cases, when a finite end maps to a wall, the derivative at the end is contained in the wall. We call such spine almost-transverse.

**Question:** Do we have deformation invariance for almost-transverse spines?

**Answer:** Yes. But the above proposition does not apply, we must restrict to skeletal curves.

Recall we have \( \Phi_i = (\text{dom, ev}_i) : \mathcal{M}^{sm}(U^{an}, P, \beta) \to \overline{\mathcal{M}}_{0, n}^{an} \times U^{an} \).

Denote \( \text{ISk} := \Phi_i^!(\overline{\text{Sk}}(M_{0, n}) \times \text{Sk}(U)) \), consisting of skeletal curves.

**Proposition (toric tail almost-transverse):** Let \( N \subset \mathcal{M}^{sm}(U^{an}, P, \beta)_\Theta \cap \text{ISk} \) be an open subspace \( f \notin N \), if \( f^{\text{torp}}(p_s) \in \sigma \) for some polyhedral cell \( \sigma \subset \text{Wall} \), then the linear span of \( \sigma \) contains \( P_e \). In this case, the subspace satisfying the toric tail condition \( N_{\text{tail}} \subset N \) is a union of connected components.

**Proof:** Openness follows again from toric tail condition (2). For closedness, let \( f \in N \) be a point in the closure. Since \( N \subset \text{ISk} \) is open, the restriction of \( \Phi_i \) to \( N_{\text{tail}} \) is open by the smoothness theorem. So we can find a net \( (f_\lambda)_{\lambda \in \Lambda} \) in \( N_{\text{tail}} \) converging to \( f \) st. \( f_\lambda^{\text{torp}} \) has transverse spine for all \( \lambda \in \Lambda \).
Let $I_{\lambda}$ denote the interval in the spine of $f_{\lambda}$ connecting $\overline{P}_s$ and the nearest branching point.

Toric tail condition (4) $\Rightarrow f_{\lambda}^{\text{trop}}$ contracts the leg $L_s$, and is balanced on $I_{\lambda} \cup L_e$ with derivative $P_e$.

If in the limit $f_{\lambda}^{\text{trop}}(P_s) \in$ some cell $\sigma \subset \text{Wall}$, our assumption says that $\langle \sigma \rangle \not\in P_e$. Then the transversality of the spine of $f_{\lambda}$ implies that $f_{\lambda}^{\text{trop}}(I_{\lambda} \cup L_s \cup L_e)$ does not meet any such $\sigma$. Therefore, by the continuity of tropicalization, for $\lambda$ sufficiently big, the distance between $\overline{P}_s$ and $(f_{\lambda}^{\text{trop}})^{-1}(\text{Wall})$ has a positive lower bound. This will prevent that any twig slides through $I_{\lambda}$ and reaches $\overline{P}_s$ in the limit. So the limit $f$ still satisfies the toric tail condition, completing the proof.

**Remark:** This proposition implies the deformation invariance for "almost transverse" spines. It is necessary for the proof of the associativity of the mirror algebra and the proof of the wall-crossing homomorphism.

5. Gluing formula and independence on the choice of torus

**Scepticism:** Recall that we have defined the counts of curves with boundaries (associated to truncated spines) by imposing an extra regularity condition on the boundaries, namely the toric tail condition. In this way, the counts of open curves are translated into special counts of closed curves. So it is natural to have some skepticism at this point: Do the counts we define really reflect open curve counting? Or is it just false advertising?

We would like to relieve this scepticism by establishing the next important property of our counts: the gluing formula.
Idea: Roughly, the gluing formula states that we can glue two open curves along two opposite boundary components, and form a bigger concatenated curve. Then the count of the concatenated curve should be the product of the counts of the two initial curves.

\[
\# \left( \begin{array}{c}
\end{array} \right) = \# \left( \begin{array}{c}
\end{array} \right) \cdot \# \left( \begin{array}{c}
\end{array} \right)
\]

This is more convincing evidence that our counts really reflect open curve counting. It is also an essential ingredient in the proof of the associativity of the mirror algebra.

Now we establish the gluing formula in 3 steps:

Step 1: Given two spines \( S^1 = (\Gamma^1, h^1) \), \( S^2 = (\Gamma^2, h^2) \) in \( Sk(U) \).

Assume we have an internal \( \infty \) 1-valent vertex \( w^1 \) in \( S^1 \) (resp. \( w^2 \) in \( S^2 \)) with \( h^1(w^1) = h^2(w^2) \).

Then consider a spine \( \Delta \) with 3 \( \infty \) 1-valent vertices \( w^1, w^2, w \), and mapping constantly to \( h^1(w^1) = h^2(w^2) \in Sk(U) \).
So we can glue $\Delta$ to $S' \sqcup S^2$ along $w'$ and $w^2$, and form a new spine $S$ in $Sk(U)$:

$\infty$ 1-valent vertices $w'$ and $w^2$ become nodes in the new spine $S$.

**Lemma:** For any curve class $\gamma \in NE(Y, \mathbb{Z})$, we have

$$N_w(S, \gamma) = \sum_{\gamma' + \gamma^2 = \gamma} N_{w'}(S', \gamma') \cdot N_{w^2}(S^2, \gamma^2)$$

**Proof:** By passing to a big enough base field extension, this follows from a set-theoretical decomposition of the set of skeletal curves associated to $S$ to products of the sets of skeletal curves associated to $S'$ and $S^2$ respectively.

**Step 2:** Given two spines $S' = (\Gamma', h')$, $S^2 = (\Gamma^2, h^2)$ in $Sk(U)$, both transverse to $Wall \subset Sk(U)$.

Assume we have $p' \in \Gamma'$, $p^2 \in \Gamma^2$ in the interior of the edges such that $h'(p') = h^2(p^2) \in Sk(U) \setminus Wall$.

So we can glue $S'$ and $S^2$ along $p'$ and $p^2$, and obtain a new transverse spine $S$ in $Sk(U)$.

**Lemma:** For any curve class $\gamma \in NE(Y, \mathbb{Z})$, we have

$$N(S, \gamma) = \sum_{\gamma' + \gamma^2 = \gamma} N(S', \gamma') \cdot N(S^2, \gamma^2)$$
Proof: We add an $\infty$-leg $w$ to $S$ at $p$, and then deform by stretching $p$:

If we further stretch the vertical edges $e'\; e^2$ to make them of $\infty$-length and contain a node, we arrive at the gluing situation of Step 1. Note that in the stretching process all spines are transverse, so their counts do not change by deformation invariance.

**Step 3:** Given two spines $S' = (\Gamma', h')$, $S^2 = (\Gamma^2, h^2)$ in $Sk(U)$, both transverse to $W_{\text{al}} \subset Sk(U)$.

Assume we have finite 1-valent vertices $v' \in \Gamma'$ and $v^2 \in \Gamma^2$ such that $h'(v') = h^2(v^2)$, and $d_{v'} h' + d_{v^2} h^2 = 0$ (i.e. opposite derivatives).

So we can glue $S'$ and $S^2$ along $v'$ and $v^2$, and obtain a new transverse spine $S$ in $Sk(U)$.

**Theorem (Gluing formula):** For any curve class $\gamma \in NE(Y, Z)$, we have...
This is a generalization of \( N(S, \gamma) = \sum_{\gamma' + \gamma^2 = \gamma} N(S', \gamma') \cdot N(S^2, \gamma^2) \) with a more conceptual proof.

Proof: We make a small extension of \( S' \) at \( v' \) to \( \hat{S}' \) by linearity, (and of \( S^2 \) at \( v^2 \) to \( \hat{S}^2 \)).

This does not change the counts by deformation invariance.

Now observe that \( \hat{S}' \sqcup \hat{S}^2 = S \sqcup \hat{L} \), a small straight spine

Apply Step 2 to both sides of the above equality, we obtain

\[
\sum_{\gamma' + \gamma^2 = \gamma} N(\hat{S}', \gamma') \cdot N(\hat{S}^2, \gamma^2) = \sum_{\beta' + \beta^2 = \gamma} N(S, \beta') \cdot N(L, \beta^2)
\]

We can explicitly compute that \( N(L, \beta^2) = \begin{cases} 1 & \text{if } \beta^2 = 0 \\ 0 & \text{otherwise} \end{cases} \) substitute \( \implies \) proof complete.

Similar idea can be applied to show that our counts are independent on the choice of torus \( T_m \subset U \) when we impose the toric tail condition:

Assume we have two torus embeddings \( T_m \subset U \) and \( T_m' \subset U \), leading to two different toric tail conditions \( T \) and \( T' \).

Consider a transverse spine \( S = (\Gamma, h) \) in \( Sk(U) \), with a finite 1-valent vertex \( v \).

Pick \( w \) very close to \( v \) and let \( L \) denote the restriction of the spine \( S \) to the interval \([w, v]\).

Pick any \( x \in (w, v) \). Consider the gluing \( S \sqcup L \).

By Step 2 above, we obtain

\[
\sum_{\beta + \delta = \gamma} N(S_T, \beta) \cdot N(L_T, \delta) = \sum_{\beta + \delta = \gamma} N(S_{T'}, \beta) \cdot N(L_T, \delta)
\]

Count using tail condition \( T \) at every end

Count using tail condition \( T' \) at \( v \), \( T \) at other ends
We compute explicitly that \( N(L_T, \delta) = \begin{cases} 1 & \text{if } \delta = 0 \\ 0 & \text{otherwise} \end{cases} \), same for \( N(L_T', \delta) \).

Substituting into the equality above, we obtain:

**Theorem (tail condition with varying torus):** We have \( N(S_T, Y) = N(S_T', Y) \), i.e. the count of skeletal curves is independent on the choice of torus \( T_m \subset U \).

**Remark:** Here the explicit computation for \( N(L_T', \delta) \) is more subtle than for \( N(L_T, \delta) \). We need to use a result concerning the gluing of non-archimedean polyannuli in my paper arXiv:1608.07651.

6. Structure constants and associativity of mirror algebra

Recall: we have log Calabi-Yau \( U = T_m \), snc compactification \( U \subset Y \), monoid ring \( R = \mathbb{Z}[\text{NE}(Y, Z)] \) assembling all curve classes, mirror algebra \( A \), having basis \( \text{Sk}(U, Z) \) as \( R \)-module.

Given \( P_1, \ldots, P_n \in \text{Sk}(U, Z) \), we write the product in the mirror algebra \( A \) as

\[
\Theta_{P_1} \cdots \Theta_{P_n} = \sum_{Q \in \text{Sk}(U, Z)} \sum_{\gamma \in \text{NE}(Y, Z)} \chi(P_1, \ldots, P_n, Q, \gamma) z^\gamma \theta_Q
\]

The structure constant \( \chi(P_1, \ldots, P_n, Q, \gamma) \) was defined as follows:

We had curve class \( \delta \) of the toric tail, and total class \( \beta := \gamma + \delta \).

We had \( Z = -Q \in M = \text{Sk}(U, Z) \) and tuple \( P_Z := (P_1, \ldots, P_n, Z) \).

Then we considered the moduli space \( H(P_Z, \beta) \) with marked points labeled as \( p_1, \ldots, p_n, z \).

We had natural map \( \Phi := (\text{dom}, \text{ev}_s) : H(P_Z, \beta) \to M_{n+n+2} \times U \), and a special point \( \widetilde{Q} = (\mu, Q) \in (M_{n+n+2} \times U)^{an} \).

Finally we had a subspace \( F \subset (\Phi^{an})^{-1}(\widetilde{Q}) \), which is a finite analytic space, whose length was by definition the structure constant \( \chi(P_1, \ldots, P_n, Q, \gamma) \).
Remark: Due to the choice of the specific point $\tilde{Q} \in (M_{0,n+2} \times U)^n$, the curves in $F$ responsible for structure constants, though highly generic in the algebraic sense, are in fact very special, i.e. non-transverse, from the tropical viewpoint. This was convenient for giving a quick definition of structure constants, but is impractical for proving any properties about them (e.g. associativity, finiteness, etc). We must deform the curves in $F$ into more transverse positions by perturbing the point $\tilde{Q} \in (M_{0,n+2} \times U)^n$.

Proposition: We label the marked points of metric trees in $M_{0,n+2}^{\text{trop}}$ as $p_1, \ldots, p_n, z, s$. Let $V_M \subset M_{0,n+2}^{\text{trop}}$ be the subset consisting of metric trees whose $z$-leg and $s$-leg are incident to a single $3$-valent vertex.

Consider the polyhedral subdivision $\Sigma$ of $\text{Sk}(U)$ given by $\text{Wall} \subset \text{Sk}(U)$ (here we can assume $\text{Wall} \subset \text{Sk}(U)$ to be finite polyhedral by bounding the degree of twigs via the fixed curve class $\beta$), and let $V_Q \subset \text{Sk}(U)$ be the open star of $Q$ in $\Sigma$ (i.e. the union of open cells in $\Sigma$ whose closure contains $Q$).

We have $\tilde{Q} \in V_M \times V_Q \subset M_{0,n+2}^{\text{trop}} \times \text{Sk}(U) \cong \text{Sk}(M_{0,n+2} \times U) \subset (M_{0,n+2} \times U)^n$.

Let $H(P_z, \beta)^{\text{an}}_{V_M \times V_Q}$ be the preimage of $V_M \times V_Q$ by $\Phi^{\text{an}}: H(P_z, \beta)^{\text{an}} \to (M_{0,n+2} \times U)^n$.

Let $F \subset H(P_z, \beta)^{\text{an}}_{V_M \times V_Q}$ be the subset satisfying the toric tail condition. Then $\Phi^{\text{an}}$ is finite étale on a neighborhood of $F$, whose degree gives the structure constant $\chi(P_1, \ldots, P_n, Q, y)$.

Proof: use the toric tail proposition for almost transverse spines.

Theorem: The multiplication rule given by the structure constants $\chi(P_1, \ldots, P_n, Q, y)$ is commutative and associative.

Sketch of proof: Commutativity is obvious because the definition of $\chi(P_1, \ldots, P_n, Q, y)$ is symmetric wrt $P_i$. Associativity means that the product $\theta_{P_1} \cdot \theta_{P_2} \cdots \theta_{P_n}$ does not change if we add arbitrary parentheses. Let us now sketch the proof of the equality:
\[(\Theta_{p_1} \cdot \Theta_{p_2}) \cdot \Theta_{p_3} = \Theta_{p_1} \cdot (\Theta_{p_2} \cdot \Theta_{p_3}) \text{ for all } P_1, P_2, P_3 \in Sk(U, Z)\]

Rewrite the products using the multiplication rule, it is equivalent to the following equality:

\[\sum_{R \in M} \sum_{\eta + \phi = \gamma} \chi(P_1, P_2, R, \eta) \cdot \chi(R, P_3, Q, \phi) = \chi(P_1, P_2, P_3, Q, \gamma) \quad (\star)\]

for every \(Q \in Sk(U, Z)\) and \(\gamma \in NE(Y, Z)\).

The RHS of (\star) is given by counts of skeletal curves associated to spines of the shape \(\text{shape} \) . By the above proposition, we can deform the modulus of the domain \(\text{domain}\), and we stretch the path \(L\) very long.

Then a point \(u\) near the top of \(L\) will map sufficiently close to the ray \(\overline{OR} \subset Sk(U)\), i.e. will lie inside the cone \(V_R \subset Sk(U)\) as in the above proposition for the structure constant \(\chi(P_1, P_2, R, \eta)\). So we can cut at \(u\), apply the gluing formula, and obtain the LHS of equality (\star).

Similarly, given two spines responsible for the product \(\chi(P_1, P_2, R, \eta) \cdot \chi(R, P_3, Q, \phi)\) in the LHS of (\star), we can glue them to form a highly stretched spine as above that is responsible for the RHS. This completes the proof of associativity.

7. Convexity and finiteness

Another important property of the structure constants \(\chi(P_1, \ldots, P_n, Q, \gamma)\) is the following convexity property:
Theorem (Convexity): Let $F$ be a (Cartier) divisor on $Y$. Let $f : [D, (P_1, \ldots, P_n)] \to Y^\an$ be an analytic disk (in general position) responsible for the structure constant $\chi(P_1, \ldots, P_n, Q, \gamma)$. The following hold:

1. We have $\sum F^\tr(P_i) - F^\tr(Q) = F \cdot \gamma - \deg(F^\an|_{D^0})$

2. Assume $F$ is nef and $-F|_U$ is effective, then $F^\tr(Q) \leq \sum F^\tr(P_i)$

3. Assume $F$ is ample and $-F|_U$ is effective, then the above inequality is an equality if and only if $f(D^0)$ lies in the torus $T^\an_M \subset U^\an$.

Proof: computation using semistable models of curves.

The convexity theorem implies the following finiteness result:

Finiteness result I: Given $P_i, \ldots, P_n \in Sk(U, \mathbb{Z})$, there are at most finitely many pairs $(Q, \gamma)$, $Q \in Sk(U, \mathbb{Z})$, $\gamma \in NE(Y, \mathbb{Z})$ such that $\chi(P_1, \ldots, P_n, Q, \gamma) \neq 0$.

Proof: Since $U$ is affine, $\exists$ regular functions $x_1, \ldots, x_i$ on $U$ s.t. the set $
\{ b \in Sk(U) | |x_i(b)| \leq c, \forall i \}$ 

is bounded for any $c \in \mathbb{R}$.

If $\chi(P_1, \ldots, P_n, Q, \gamma) \neq 0$, by the Convexity Theorem (3), we have

$|x_i(Q)| \leq \sum_j |x_j(P_j)|, \forall i$

Thus given $P_1, \ldots, P_n$, $\exists$ at most finitely many $Q$ s.t. $\chi(P_1, \ldots, P_n, Q, \gamma) \neq 0$ for some $\gamma$.

Next we bound $\gamma$: $U$ affine $\Rightarrow \exists$ ample divisor $F$ on $Y$ with $-F|_U$ effective.

Now apply Convexity Theorem (1), and obtain

$F \cdot \gamma = \sum F^\tr(P_i) - F^\tr(Q) + \deg(F^\an|_{D^0}) \leq \sum F^\tr(P_i) - F^\tr(Q)$.

This bounds $\gamma$ by the ampleness of $F$.

Remark: 1) Finiteness result I implies that the two sums in the multiplication rule
\[ \theta_{p_1} \cdots \theta_{p_n} = \sum_{\lambda \in \text{sk}(U, Z)} \sum_{\gamma \in \text{NE}(Y, Z)} \chi(p_1, \ldots, p_n; \lambda, \gamma) z^\lambda \theta_{\gamma} \]

are finite sums, so it gives an $R$-algebra structure on the free $R$-module $A$ (instead of just some formal algebra structure).

2) We have the following stronger:

**Finiteness result II:** The mirror algebra is a finitely generated $R$-algebra.

For its proof, we need to resort to the equivariant boundary torus action on the mirror algebra.

8. Boundary torus action and finite generation

  Omitted in this lecture due to time constraints.