Cluster Algebra via Non-Archimedean Geometry

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History: On June 10, 2013, I gave the first talk in a math conference in my life. The conference was named: “Cluster Algebras and Tropical Geometry”, and was held at IRMA, Strasbourg.

I asked Bernhard Keller: What is the relation

Bernhard replied: It is not clear. So we are organizing this conference to figure out.
In November 2014, comes the preprint Gross-Hacking-Keel-Kontsevich «Canonical bases for cluster algebras»

![Diagram: Cluster Algebras to Scattering Diagrams via Canonical bases]

These are combinatorial gadgets. They have the flavor of tropical geometry, but it is not clear what exactly are the geometric objects behind these combinatorial gadgets.

GHKK wrote: Morally, they should be tropicalizations of holomorphic disks. But how to go from “Morally” ⟷ “Scientifically”?

Lacking a precise geometric interpretation, several things in GHKK remain conjectural: feels right but unable to prove within the combinatorial framework.

Goal of this talk: Answer the above question:

- Find the right geometric object
- Establish the relation with scattering diagram/broken lines
- Prove GHKK conjectures
Direct algebro-geometric approach w/o passing through combinatorial gadgets

Non-archimedean analytic curves in cluster varieties

Non-archimedean analytic curve

Idea: non-archimedean analytic curve $\to$ skeleton $\to$ broken lines $\to$ scattering diagram

Plan of the more technical part of the talk:

1. Log Calabi-Yau varieties & Essential skeletons
2. Skeletal curves: a key notion in the theory
3. Naive counts of skeletal curves
4. Canonical scattering diagram
5. Application to cluster algebra:
   - Positivity of the Laurent phenomenon
   - GHKK broken-line convexity conjecture
   - Removal of GHKK EGM assumption
   - Independence of choice of seed conjecture
   - Geometric explanation of the change of scattering diagram under mutation
1. Log Calabi-Yau varieties and essential skeletons

Fix $k$ any non-archimedean field, e.g. $\mathbb{C}((t))$, $\mathbb{Q}_p$, $\mathbb{C}$ trivial valuation

Recall 1: A algebraic variety $X/k$

Berkovich analytification $\xrightarrow{}$ $k$-analytic space $X^{\text{an}}$

\[
X^{\text{an}} = \left\{ (\mathfrak{p}, v) \mid \mathfrak{p} \in X \text{ scheme point, } v \text{ is an absolute value on the residue field } K(\mathfrak{p}) \right\}
\]

+ weak topology + sheaf of analytic functions

Recall 2: A pluricanonical form $\omega \in \Gamma \left( \bigwedge^d \Omega_X \otimes e \right)$

\[\{\text{Temkin Kähler seminorm}\]

upper semicontinuous function $\|\omega\| : X^{\text{an}} \rightarrow \mathbb{R}_{\geq 0}$

We define $\text{Sk}(\omega) \subset X^{\text{an}}$, maximal locus of $\|\omega\|$

Polyhedral subset
$\text{Sk}(X) := \bigcup \text{Sk}(w) \subset X^{\text{an}}$

the essential skeleton

called all nonzero log pluricanonical form $w$
of $X$ (studied by Kontsevich–Soibelman, Mustață–Nicaise, Brown–Mazzon, Temkin)
in various levels of generality.

Example: \( X = G_m^n \) \( \text{Sk}(X) \cong \mathbb{R}^n \subset (G_m^n)^{\text{an}} \)
- $X$ elliptic curve with bad reduction
  \( \text{Sk}(X) \cong S^1 \subset X^{\text{an}} \)
- $X$ $K3$ surface with maximal degeneration
  \( \text{Sk}(X) \cong S^2 \subset X^{\text{an}} \)

Def: A smooth algebraic variety $U/k$ is called log Calabi–Yau if all log pluricanonical bundles are trivialized by the tensor power $w^\otimes n$ of a volume form $w$.

Example: If $U$ contains a Zariski dense open algebraic torus $G_m^n$ (e.g. the case of cluster variety), then $\text{Sk}(U) = \text{Sk}(G_m^n) \cong \mathbb{R}^n$.

2. Skeletal curves: a key notion in the theory.

Idea: Consider a curve $C$ in a log Calabi–Yau variety $U^{\text{an}}$. 
If \( \dim U \geq 2 \), by dimensional reason, \( C \) has no chance to meet \( \text{Sk}(U) \).

"Valuations on the generic point"

But we can let \( C \) touch \( \text{Sk}(U) \) if we allow \( C \) to be defined over a big non-archimedean field extension \( k \subset k' \).

**Surprise:** As soon as some \( k \)-point of \( C \) touches \( \text{Sk}(U) \), the whole skeleton of \( C \) must lie in \( \text{Sk}(U) \).

**Precise statement:** Fix \( U \) log \( (Y/k) \), vol form \( w \), \( U \subset Y \) snc compactification, \( D := Y \setminus U \) \( \text{D}^{\text{ess}} \subset D \) essential divisors (i.e. where \( w \) has a pole)

**Theorem (KY):** Let \( k \subset k' \) non-archimedean field extension.

\( C \) rational nodal curve \( /k \)

\[ f: C_{k'} \to Y_{k'}^{\text{an}} \]

\( k' \)-an map s.t. \( f^{-1}(D) = f^{-1}(\text{D}^{\text{ess}}) = \sum m_i P_i \), \( P_i \in C(k) \)

\[ f_Y: C' \to Y_{k'}^{\text{an}} \to Y_{k}^{\text{an}} \]

composition

If \( f_Y(x) \in \text{Sk}(U) \) for some \( x \in C(k) \subset C_{k'} \),

\( \bigcap_{C \setminus \{P_i\}} \)

then \( f_Y(\text{Sk}(C)) \subset \text{Sk}(U) \) i.e. the whole skeleton of the curve lies in \( \text{Sk}(U) \).

the convex hull of all \( P_i \) in \( C_{k'}^{\text{an}} \)

**Def:** We call such \( f \) skeletal curves
Advantage of skeletal curve: They have canonical tropicalization

\[ \Gamma := \text{Sk}(C^0) \xrightarrow{f_Y} \text{Sk}(U) \]

independent of any choice of retraction \( U^\text{an} \to \text{Sk}(U) \)

\( \text{e.g. different minimal model } U \subset Y \) gives different retraction.

But for skeletal curves, the compactification \( U \subset Y \) does not matter.

Def: We call this the spine associated to the skeletal curve.

Question: The skeletal curves seem so nice, but where do they come from in practice?

Natural source of skeletal curves:

From now on: \( k \) char = 0, trivial valuation, e.g. \( k = \mathbb{C} \).
\( U \log(CY/k), U \subset Y \) snc compactification, \( D := Y \setminus U \)

\( \text{Sk}(U, Z) \subset \text{Sk}(U) \subset U^\text{an} \)

\[ \{0\} \cup \{mv \mid \text{v is a divisorial valuation on } k(U), m \in \mathbb{N}_{>0}\} \]

Given an \( n \)-tuple \( P = (P_1, \ldots, P_n) \) with \( P_i \in \text{Sk}(U, Z) \) and \( P \in \text{NE}(Y, Z) \),
write \( P_i = m_i v_i \) for all \( P_i \neq 0 \).
Up to modifying the Snc compactification $U \subset Y$, we can assume that each $U_i$ has divisorial center $D_i \subset D$.

Define moduli stack $M(U, P, \beta) := \{n$-pointed rational stable maps $f : (C, p_1, \ldots, p_n) \to Y$ of class $\beta \}$ s.t. $p_i$ meets $D_i \circ$ with order $m_i$ for all $P_i \neq 0$, no other intersection with $D$

Now we can evaluate at the “interior marked points”:
Fix $i$ with $P_i = 0$, consider:

$$\overline{\Phi}_i := (st, ev_i) : M(U, P, \beta)^{an} \to \overline{M}_{0,n} \times U^{an}$$

Stabilization evaluation closure

Theorem: $\overline{\Phi}_i$ over $\overline{Sk(M_{0,n})} \times Sk(U)$ has finite fibers.
and the preimage consists of skeletal curves.

3. Naive counts of skeletal curves with given spine

$\forall f : (C, p_1, \ldots, p_n) \to Y^{an} \in \overline{\Phi}_i^{-1}(\overline{Sk(M_{0,n})} \times Sk(U))$

Theorem above $\Rightarrow$ its spine $\Gamma \xrightarrow{h} Sk(U)$ is well-defined.
Conversely, given any abstract spine $\Gamma \xrightarrow{h} Sk(U)$, and curve class $\beta \in NE(Y, \mathbb{Z})$
We want to count all skeletal curves (of class $\beta$) giving rise to this spine $h$.

Goal: Define this count $N(h, \beta)$.

Construction: $\Gamma \xrightarrow{h} Sk(U)$, piecewise affine
$\Gamma$ has $n$ legs $v_1, \ldots, v_n$ with weight vectors
$P_1, \ldots, P_n \in Sk(U, \mathbb{Z})$
Recall: $\Phi_i = (st, ev_i): M(U, P, \beta)^{an} \rightarrow \bar{M}_{g,n}^{an} \times U^{an}$
Let $F_i(h, \beta) \subset \Phi_i^{-1}(\Gamma, h(v_i))$ be the subspace consisting of maps whose spine is equal to $h$.
We define $N_i(h, \beta) := \text{length}(F_i(h, \beta))$
Intuitively, $N_i(h, \beta)$ counts "closed curves" with given spine.

More generally, we can define counts of "open curves" associated to truncated spines, i.e. spine with finite endpoints:

Construction: Given $h$ truncated spine, $\gamma \in \text{NE}(Y)$

extend the finite ends to $\infty \Rightarrow \hat{h}$ extended spine, $\hat{\gamma}$ extended curve class

Assume $T \subset U$ Zar open torus, let

$F_i(h, \gamma) \subset F_i(\hat{h}, \hat{\gamma})$

be the subspace satisfying the "toric tail condition":

We consider the disk associated to each added tail (i.e. extension of a finite end).

We define $N_i(h, \gamma) := \text{length} F_i(h, \gamma)$
Theorem: The count \( N_i(h, y) \) is independent of the choice of the torus \( T \subset U \), nor of the internal marked point \( i \).

Now we are ready to construct a scattering diagram using the counts above.

4. Canonical scattering diagram

Fix \( T \subset U \subset Y \), denote \( N := \text{Hom}(M, \mathbb{Z}) \).

Definition: Given hyperplane \( n^+ \subset M_{\mathbb{R}} \) (\( n \in N \)), \( x \in n^+ \) generic, \( v, w \in M \setminus n^+ \), and curve class \( \alpha \in \text{NE}(Y, \mathbb{Z}) \), let \( V_{x,v,w} \) be the infinitesimal spine bending at \( x \), with incoming direction \( w \) and outgoing direction \( v \).

\[ \cdots \] associated count of analytic curves \( N(V_{x,v,w}, \alpha) \).

Definition: For any \( x \in n^+ \subset M_{\mathbb{R}} \) generic, we define the wall-crossing transformation:

\[
\Psi_{x,n}(z^v) = \begin{cases} 
\sum_{\substack{w \in M, \langle n, w \rangle > 0 \alpha \in \text{NE}(Y) \atop \alpha \in \text{NE}(Y)}} N(V_{x,v,w}, \alpha) z^\alpha z^w & \text{for } \forall v \in M \text{ with } \langle n, v \rangle > 0. \\
z^v & \text{for } \forall v \in M \text{ with } \langle n, v \rangle = 0.
\end{cases}
\]

For convergence: Fix any strictly convex toric monoid \( Q \supset \text{NE}(Y) \).

Let \( \hat{R} \) be the completion of \( \mathbb{Z}[Q \otimes M] \) wrt maximal monomial ideal.
Theorem: $\Psi_{x,n}$ extends to an automorphism of $\text{Frac} \hat{R}$.

Moreover, $\exists f_x \in \hat{R}$ st. $\forall v \in M$ with $\langle n,v \rangle > 0$, we have

$$\Psi_{x,n}(z^v) = z^v f_x$$

called scattering function

Rem: $\Psi_{x,n}$ does not give rise to an automorphism of $\hat{R}$ because $f_x$ may not be invertible. We make it invertible by ignoring all curve classes.

Assumption: All bends of $V_{x,v,w}$ for $N(V_{x,v,w}, \alpha) \neq 0$ lie in a strictly convex toric monoid $P \subseteq M$.

Notation: $J \subseteq \mathbb{Z}[P]$ maximal monomial ideal,

$$\hat{\mathbb{Z}}^0 := J\text{-adic completion of } \mathbb{Z}[P]$$

$$\hat{\mathbb{Z}} := \hat{\mathbb{Z}}^0 \otimes_{\mathbb{Z}[P]} \mathbb{Z}[M]$$

Proposition: Under the quotient $\mathbb{Z}[Q \oplus P] \to \mathbb{Z}[P]$

$$f_x \in \hat{R} \mapsto \bar{f}_x \in \hat{\mathbb{Z}}$$

$$\Psi_{x,n} \mapsto \bar{\Psi}_{x,n} \in \text{Aut}(\hat{\mathbb{Z}})$$

Theorem: The scattering diagram given by $\bar{\Psi}_{x,n}$ is consistent in the sense of Kontsevich–Soibelman–Gross–Siebert.

I.e. A general loop $\ell : S^1 \to M_{\mathbb{R}}$, the composition of wall-crossing automorphisms $\bar{\Psi}_{x,n}$ along $\ell$ is the identity on $\hat{\mathbb{Z}}$.

5. Application to cluster algebra
Cluster data: Lattice $M$ with a unimodular skew-symmetric form holds in the principle coeff case.

$S'$ a basis of $M$, $S \subset S'$ a subset.

$\mapsto$ a seed for a skew-symmetric cluster algebra of geometric type

$S \leftrightarrow$ unfrozen variables $S' \setminus S \leftrightarrow$ frozen variables

$\mapsto$ A Fock-Goncharov A-type cluster variety (gluing of tori via cluster mutations)

Let $A^u := \Gamma(\mathcal{A}, \mathcal{O}_\mathcal{A})$, called upper cluster algebra.

Now apply our constructions to $U := \text{Spec } A^u$, the cluster variety.

Comparison theorem: The scattering diagram given by counting non-archimedean analytic curves is equivalent to the scattering diagram of Gross-Hacking-Keel-Kontsevich. (Same holds for X-type cluster algebras)

Consequence: (1) Our naïve counts are always non-negative integers

$\Rightarrow$ much more conceptual proof of the positivity of the coefficients of the scattering functions (as well as of the structure constants)

$\Rightarrow$ Positivity in the Laurent phenomenon

(2) Proof of GHKK broken-line convexity conjecture

(3) Removal of GHKK’s EGM assumption.

(4) Shows the mirror algebra is independent of choice of cluster structure, also conjectured by GHKK.

(5) Geometric explanation of the change of scattering diagram under mutation.
Final remark:

Using non-archimedean enumerative geometry, one can actually bypass completely scattering diagrams, and achieve a direct canonical construction (intrinsic to the cluster variety, independent of the cluster structure).

More details: My mini-course at IHES last month, available on YouTube.