CENTRAL LIMIT THEOREM FOR LINEAR GROUPS

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Abstract. We prove a central limit theorem for random walks with finite variance on linear groups.

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1. Introduction

1.1. **Central limit theorem for linear groups.** Let $V = \mathbb{R}^d$, $G = \mathrm{GL}(V)$ and μ be a Borel probability measure on G. We fix a norm $\|.\|$ on V. For $n \geq 1$, we denote by μ^{*n} the n^{th}-convolution power $\mu * \cdots * \mu$. We assume that the first moment $\int_G \log N(g)\, \mathrm{d}\mu(g)$ is finite, where $N(g) = \max(\|g\|, \|g^{-1}\|)$. We denote by λ_1 the first Lyapunov exponent of μ, i.e.

$$\lambda_1 := \lim_{n\to\infty} \tfrac{1}{n} \int_G \log \|g\|\, \mathrm{d}\mu^{*n}(g). \tag{1.1}$$

Let $g_1, \dots, g_n, \dots$ be random elements of G chosen independantly with law μ. The Furstenberg law of large numbers describes the behavior of the random variables $\log \|g_n \cdots g_1\|$. It states that, almost surely,

$$\lim_{n\to\infty} \tfrac{1}{n} \log \|g_n \cdots g_1\| = \lambda_1. \tag{1.2}$$

In this paper we will prove that, under suitable conditions, the variables $\log \|g_n \cdots g_1\|$ satisfy a central limit theorem (CLT) i.e. that the renormalized variables $\frac{\log \|g_n \cdots g_1\| - n\lambda_1}{\sqrt{n}}$ converge in law to a non degenerate Gaussian variable.

Let Γ_μ be the semigroup spanned by the support of μ. We say that Γ_μ acts strongly irreducibly on V if no proper finite union of vector subspaces of V is Γ_μ-invariant.

Theorem 1.1. *Let $V = \mathbb{R}^d$, $G = \mathrm{GL}(V)$ and μ be a Borel probability measure on G such that Γ_μ has unbounded image in $\mathrm{PGL}(V)$, Γ_μ acts strongly irreducibly on V, and the second moment $\int_G (\log N(g))^2 \, \mathrm{d}\mu(g)$ is finite. Let λ_1 be the first Lyapunov exponent of μ. Then there exists $\Phi > 0$ such that, for any bounded continuous function F on $\mathbb{R}$, one has*

$$\lim_{n\to\infty} \int_G F\left(\tfrac{\log \|g\| - n\lambda_1}{\sqrt{n}}\right) \mathrm{d}\mu^{*n}(g) = \int_{\mathbb{R}} F(s) \tfrac{e^{-\frac{s^2}{2\Phi}}}{\sqrt{2\pi\Phi}} \, \mathrm{d}s. \tag{1.3}$$

Remarks 1.2. - We will see that under the same assumptions the variables $\log \|g_n \cdots g_1\|$ also satisfy a law of the iterated logarithm (LIL) i.e. almost surely, the set of cluster points of the sequence $\frac{\log \|g_n \cdots g_1\| - n\lambda_1}{\sqrt{2\Phi\, n \log\log n}}$ is equal to the interval [-1,1].

- According to a result of Furstenberg, when moreover Γ_μ is included in the group $\mathrm{SL}(V)$, the first Lyapunov exponent is positive: $\lambda_1 > 0$.

- For every non-zero v in V and f in V^*, the variables $\log \|g_n \cdots g_1 v\|$ and $\log |f(g_n \cdots g_1 v)|$ also satisfy the CLT and the LIL.

- Such a central limit theorem is not always true when the action of Γ_μ is only assumed to be irreducible: in this case the variables $\frac{\log \|g_n \cdots g_1\| - n\lambda_1}{\sqrt{n}}$ still converge in law but the limit is not always a Gaussian variable (see Example 4.15).

- The assumption "Γ_μ has unbounded image in $\mathrm{PGL}(V)$" is used only to ensure that the limit Gaussian law is non-degenerate. Indeed, one can check that the limit Gaussian law is degenerate if and only if there exist a constant $t_0 > 0$ and a compact subgroup K of $\mathrm{GL}(V)$ such that the support of μ is included in the set $t_0 K := \{t_0 k \mid k \in K\}$.

- We will deduce easily a multidimensional version of this CLT (Theorem 4.11) and interpret it as a CLT for real semisimple groups (Theorem 4.16), generalizing Goldsheid and Guivarc'h CLT in [23]. Most

of our results are true over any local field $\mathbb{K}$ with no changes in the proofs.

1.2. **Previous results.** Let us give a historical perspective about this theorem. The existence of such a "non-commutative CLT" was first guessed by Bellman in [3]. Such a theorem has first been proved by Furstenberg and Kesten in [21] for semigroups of positive matrices under an $\mathrm{L}^{2+\varepsilon}$ assumption for some $\varepsilon > 0$. It was then extended by Lepage in [39] for more general semigroups when the law has a finite exponential moment i.e. when there exists $\alpha > 0$ such that $\int_G N(g)^\alpha \, \mathrm{d}\mu(g) < \infty$. Thanks to later works of Guivarc'h and Raugi in [31] and Goldsheid and Margulis in [24] the assumptions in the Lepage theorem were clarified: the sole remaining but still unwanted assumption was that μ had a finite exponential moment.

Hence the purpose of our Theorem 1.1 is to replace this finite exponential moment assumption by a finite second moment assumption. Such a finite second moment assumption is optimal.

Partial results have been obtained recently in this direction. Tutubalin in [45] has proved Theorem 1.1 when the law μ is assumed to have a density. Jan in his thesis (see [36]) has extended the Lepage theorem under the assumption that all the p-moments of μ are finite. Hennion in [34] has proved Theorem 1.1 in the case of semigroups of positive matrices.

There exist a few books and surveys ([14], [20] or [10]) about this theory of "products of random matrices". This theory has had recently nice applications to the study of discrete subgroups of Lie groups (as in [28], [15] or [6]). These applications motivated our interest in a better understanding of this CLT.

1.3. **Other Central Limit Theorems.** The method we introduce in this paper is very flexible since it does not rely on a spectral gap property. In the forthcoming paper [9], we will adapt this method to prove the CLT in other situations where the CLT is only known under a finite exponential moment assumption:
- The CLT for free groups due to Sawyer-Steger in [41] and Ledrappier in [38],
- The CLT for Gromov hyperbolic groups due to Bjorklund in [11].

1.4. **Strategy.** We explain now in few words the strategy of the proof of our central limit theorem 1.1. We want to prove the central limit theorem for the random variables $\kappa(g_n \cdots g_1)$ where the quantity

$$\kappa(g) := \log \|g\| \tag{1.4}$$

controls the size of the element g in G. Let $X := \mathbb{P}(V)$ be the projective space of the vector space $V := \mathbb{R}^d$. Since this function κ on G is closely related to the "norm cocycle" $\sigma : G \times X \to \mathbb{R}$ given by

$$\sigma(g, x) := \log \frac{\|gv\|}{\|v\|}, \tag{1.5}$$

for g in G and $x = \mathbb{R}v$ in $\mathbb{P}(V)$, we are reduced to prove, for every x in X, a central limit theorem for the random variables $\sigma(g_n \cdots g_1, x)$.

We will follow Gordin's method. This method has been introduced in [25] and [26] and has been often used since then, see for instance [37], [11]. See also [18] and [12, Appendix] for a survey of this method and [12, Section 2.4] for the use of this method in order to prove a CLT and an invariance principle in the context of products of independant random matrices.

Following Gordin's method means that, we will replace, adding a suitable coboundary, this cocycle σ by another cocycle σ_0 for which the "expected increase" is constant i.e. such that

$$\int_G \sigma_0(g, x)\, \mathrm{d}\mu(g) = \lambda_1$$

for all x in X. This will allow us to use the classical central limit theorem for martingales due to Brown in [17]. In order to find this cocycle σ_0, we have to find a continuous function $\psi \in \mathcal{C}^0(X)$ which satisfies the following cohomological equation

$$\varphi = \psi - P_\mu \psi + \lambda_1\,, \tag{1.6}$$

where $P_\mu \psi$ is the averaged function

$$P_\mu \psi : x \mapsto \textstyle\int_G \psi(gx)\, \mathrm{d}\mu(g)$$

and where $\varphi \in \mathcal{C}^0(X)$ is the expected increase of the cocycle σ

$$\varphi : x \mapsto \textstyle\int_G \sigma(g, x)\, \mathrm{d}\mu(g). \tag{1.7}$$

The classical strategy to solve this cohomological equation relies on spectral properties of this operator P_μ. These spectral properties might not be valid under a finite second moment assumption. This is where our strategy differs from the classical strategy: we solve this cohomological equation by giving an explicit formula for the solution ψ in terms of the $\check{\mu}$-stationary measure ν^* on the dual projective space $\mathbb{P}(V^*)$, where $\check{\mu}$ is the image of μ by $g \mapsto g^{-1}$. This formula is

$$\psi(x) = \textstyle\int_{\mathbb{P}(V^*)} \log \delta(x, y)\, \mathrm{d}\nu^*(y), \tag{1.8}$$

where $\delta(x, y) = \frac{|f(v)|}{\|f\|\,\|v\|}$, for $x = \mathbb{R}v$ in $\mathbb{P}(V)$ and $y = \mathbb{R}f$ in $\mathbb{P}(V^*)$ (Proposition 4.9).

The main issue is to check that this integral is finite, i.e. that the stationary measure ν^* is log-regular, when the second moment of μ is finite (Proposition 4.5).

Let us recall the Hsu-Robbins theorem which seems at a first glance unrelated. This theorem is a strengthening of the classical law of large numbers for centered square-integrable independent identically distributed random real variables $(\varphi_n)_{n\geq 1}$. This theorem tells us that the averages $\frac{1}{n}(\varphi_1 + \cdots + \varphi_n)$ converge completely to 0, i.e. that, for all $\varepsilon > 0$, the following series converge :

$$\sum_{n\geq 1} \mathbb{P}(\tfrac{1}{n}|\varphi_1 + \cdots + \varphi_n| > \varepsilon) < \infty \,. \tag{1.9}$$

The key point to prove the log-regularity of the stationary measure ν^* is to prove an analogue of the Hsu-Robbins theorem for martingales under a suitable condition of domination by a square-integrable function (Theorem 2.2) and to deduce from it another analogue of the Hsu-Robbins theorem for the Furstenberg law of large numbers (Proposition 4.1).

Another important ingredient in the proof of the log-regularity of ν^* is the simplicity of the first Lyapounov exponent due to Guivarc'h in [27] and [31].

1.5. **Plan.** In Chapter 2, we prove the complete convergence in the law of large numbers for martingales with square-integrable increments and we recall the central limit theorem for these martingales with square-integrable increments.

In Chapter 3, we prove a large deviations estimate in the Breiman law of large numbers for functions over a Markov-Feller chain, we deduce the complete convergence in the law of large numbers for square-integrable cocycles over random walks and the central limit theorem when the cocycle is centerable.

In Chapter 4, we prove successively the complete convergence in the Furstenberg law of large numbers, the log-regularity of the corresponding stationary measure on the projective space, the centerability of the norm cocycle, and the central limit theorem 1.1. We end this chapter by the multidimensional version of this central limit theorem.

2. Limit theorems for martingales

We collect in this chapter the limit theorems for martingales that we will need in Chapter 3.

2.1. **Complete convergence for martingales.**

In this section we prove the complete convergence in the law of large numbers for martingales.

Let $(\Omega, \mathcal{B}, \mathbb{P})$ be a probability space. We first recall that a sequence X_n of random variables converges completely to X_∞, if, for all $\varepsilon > 0$, $\sum_{n\geq 1} \mathbb{P}(|X_n - X_\infty| \geq \varepsilon) < \infty$. By Borel-Cantelli Lemma, complete convergence implies almost sure convergence. We recall now the following classical result due to Baum and Katz in [2].

Fact 2.1. *Let $p \geq 1$, let $(\varphi_n)_{n\geq 1}$ be independant identically distributed real random variables and $S_n = \varphi_1 + \cdots + \varphi_n$. The following statements are equivalent:*
(i) $\mathbb{E}|\varphi_1|^p < \infty$ *and* $\mathbb{E}(\varphi_1) = 0$,
(ii) $\sum_{n\geq 1} n^{p-2}\mathbb{P}(|S_n| \geq n\varepsilon) < \infty$, *for all* $\varepsilon > 0$.

When $p = 2$ the implication $(i) \Rightarrow (ii)$ is due to Hsu-Robbins [35] and the converse is due to Erdös [19]. In this case, condition (ii) means that the sequence $\frac{1}{n}S_n$ converges completely towards 0.

When $p = 1$ this fact is due to Spitzer [42].

Our aim is to prove the following generalization of Baum-Katz theorem to martingales. Let $\mathcal{B}_0 \subset \cdots \subset \mathcal{B}_n \subset \cdots$ be sub-σ-algebras of $\mathcal{B}$. We recall that a *martingale difference* is a sequence $(\varphi_n)_{n\geq 1}$ of integrable random variables on Ω such that $\mathbb{E}(\varphi_n \mid \mathcal{B}_{n-1}) = 0$ for all $n \geq 1$.

Theorem 2.2. *Let $p > 1$, let $(\varphi_n)_{n\geq 1}$ be a martingale difference and $S_n := \varphi_1 + \cdots + \varphi_n$ the corresponding martingale. We assume that there exists a positive function φ in $\mathrm{L}^p(\Omega)$ such that, for $n \geq 1$, $t > 0$,*

$$\mathbb{E}(\mathbf{1}_{\{|\varphi_n|>t\}} \mid \mathcal{B}_{n-1}) \leq \mathbb{P}(\{\varphi > t\}) \text{ almost surely.} \tag{2.1}$$

Then there exist constants $C_n = C_n(p, \varepsilon, \varphi)$ such that, for $n \geq 1$, $\varepsilon > 0$,

$$\mathbb{P}(|S_n| > n\varepsilon) \leq C_n \text{ and } \sum_{n\geq 1} n^{p-2} C_n < \infty \,. \tag{2.2}$$

The fact that the constants C_n are controled by the dominating function φ will be important in our applications. A related theorem was stated in [44] for $p > 2$. The extension to the case $p = 2$ is crucial for our applications. We stated our result for $p > 1$ since the proof is not very different when $p = 2$.

Proof. Our proof combines the original proof of Baum-Katz theorem with Burkholder inequality. Since $p > 1$, we pick $\gamma < 1$ such that

$\gamma > \frac{p+1}{2p}$. We set, for $k \leq n$,

$$\varphi_{n,k} := \varphi_k \, \mathbf{1}_{\{|\varphi_k| \leq n^\gamma\}} \text{ and } T_n := \sum_{1 \leq k \leq n} \varphi_{n,k} \, . \tag{2.3}$$

In order to lighten the calculations we also set

$$\overline{\varphi}_{n,k} := \varphi_{n,k} - \mathbb{E}(\varphi_{n,k} \mid \mathcal{B}_{k-1}) \text{ and } \overline{T}_n := \sum_{1 \leq k \leq n} \overline{\varphi}_{n,k} \tag{2.4}$$

so that, for all $n \geq 1$, the finite sequence $(\overline{\varphi}_{n,k})_{1 \leq k \leq n}$ is also a difference martingale. We can assume $\varepsilon = 3$. We will decompose the event $A_n := \{|S_n| > 3n\}$ into four pieces

$$A_n \subset A_{1,n} \cup A_{2,n} \cup A_{3,n} \cup A_{4,n}. \tag{2.5}$$

The events $A_{i,n}$ are given by

$A_{1,n} := \{\text{there exists } k \leq n \text{ such that } |\varphi_k| > n\}$

$A_{2,n} := \{\text{there exist } k_1 < k_2 \leq n \text{ such that } |\varphi_{k_1}| > n^\gamma \, , \, |\varphi_{k_2}| > n^\gamma\}$

$A_{3,n} := \{|T_n - \overline{T}_n| > n\}$

$A_{4,n} := \{|\overline{T}_n| > n\}$

The inclusion (2.5) is satisfied since, when none of the four events $A_{i,n}$ is satisfied, one has $|S_n| \leq 3n$. We will find, for each piece $A_{i,n}$, a constant $C_{i,n} = C_{i,n}(p, \varepsilon, \varphi)$ such that $\mathbb{P}(A_{i,n}) \leq C_{i,n}$ and $\sum_{n \geq 1} n^{p-2} C_{i,n} < \infty$.

First piece. One computes, using the domination (2.1),

$$\mathbb{P}(A_{1,n}) \leq C_{1,n} := n \, \mathbb{P}(\varphi > n)$$

and

$$\sum_{n \geq 1} n^{p-2} C_{1,n} = \sum_{n \geq 1} n^{p-1} \mathbb{P}(\varphi > n) \leq \tfrac{1}{p} \, \mathbb{E}((\varphi + 1)^p)$$

which is finite since the dominating function φ is L^p-integrable.

Second piece. One computes, using the domination (2.1),

$$\mathbb{P}(A_{2,n}) \leq C_{2,n} := n^2 \, \mathbb{P}(\varphi > n^\gamma)^2$$

and, using Chebyshev's inequality,

$$\sum_{n \geq 1} n^{p-2} C_{2,n} \leq \sum_{n \geq 1} n^{p - 2\gamma p} (\mathbb{E}(\varphi^p))^2$$

which is finite since $\gamma > \frac{p+1}{2p}$.

Third piece. One bounds, remembering that the variables φ_k are martingale differences and using the domination (2.1),

$$\begin{aligned}|\mathbb{E}(\varphi_{n,k} \mid \mathcal{B}_{k-1})| &= |\mathbb{E}(\varphi_k - \varphi_{n,k} \mid \mathcal{B}_{k-1})| \\ &\leq \textstyle\int_{n^\gamma}^\infty \mathbb{P}(|\varphi_k| > t \mid \mathcal{B}_{k-1})\,\mathrm{d}t \;+\; n^\gamma \mathbb{P}(|\varphi_k| > n^\gamma \mid \mathcal{B}_{k-1}) \\ &\leq \textstyle\int_{n^\gamma}^\infty \mathbb{P}(\varphi > t)\,\mathrm{d}t \;+\; n^\gamma \mathbb{P}(\varphi > n^\gamma) = \mathbb{E}(\varphi\, \mathbf{1}_{\{\varphi > n^\gamma\}})\;,\end{aligned}$$

and this right-hand side converges to 0 when n goes to infinity since the dominating function φ is integrable. One deduces the bounds

$$\tfrac{1}{n}\,|T_n - \overline{T}_n| \leq \mathbb{E}(\varphi\, \mathbf{1}_{\{\varphi > n^\gamma\}})\;,$$

with a right-hand side also converging to 0. Hence one can find an integer $n_0 = n_0(p, \varepsilon, \varphi)$ such that, for $n \geq n_0$, the event $A_{3,n}$ is empty. We just set $C_{3,n} = 0$ when $n \geq n_0$ and $C_{3,n} = 1$ otherwise.

Fourth piece. We set $\overline{Q}_n := \sum_{1 \leq k \leq n} \overline{\varphi}_{n,k}^2$, $p_0 := \min(p, 2)$, and $M \geq 1$ to be the smallest integer such that $M \geq \frac{p}{2(1-\gamma)}$. According to the Burkholder inequality (see [32]), since $(\overline{\varphi}_{n,k})_{1 \leq k \leq n}$ is a martingale difference, there exists a constant D_M, which depends only on M, such that

$$D_M^{-1}\,\mathbb{E}(\overline{Q}_n^M) \leq \mathbb{E}(\overline{T}_n^{2M}) \leq D_M\,\mathbb{E}(\overline{Q}_n^M).$$

One computes then, using Chebyshev's inequality,

$$\mathbb{P}(A_{4,n}) \leq n^{-2M}\,\mathbb{E}(\overline{T}_n^{2M}) \leq D_M\, n^{-2M}\,\mathbb{E}(\overline{Q}_n^M). \tag{2.6}$$

We expand now $\mathbb{E}(\overline{Q}_n^M)$ as a sum of terms of the form $\mathbb{E}(\overline{\varphi}_{n,k_1}^{2q_1} \cdots \overline{\varphi}_{n,k_\ell}^{2q_\ell})$ with $1 \leq \ell \leq M$, $q_1, \ldots, q_\ell \geq 1$, $q_1 + \cdots + q_\ell = M$ and $1 \leq k_1 < \cdots < k_\ell \leq n$. Using the bounds, for $1 \leq k \leq n$ and $q \geq 1$,

$$\overline{\varphi}_{n,k}^{2q} \leq (2n^\gamma)^{2q-p_0} |\overline{\varphi}_{n,k}|^{p_0}\;,$$

and, using the domination (2.1), one bounds each term in the sum

$$\mathbb{E}(\overline{\varphi}_{n,k_1}^{2q_1} \cdots \overline{\varphi}_{n,k_\ell}^{2q_\ell}) \leq 4^M n^{2M\gamma - \ell p_0 \gamma}\,\mathbb{E}(\varphi^{p_0})^\ell\,.$$

For each value of $\ell \leq M$, the number of such terms is bounded by $M^\ell n^\ell$. Summing all these bounds, one gets, since $\gamma p_0 > \min(\frac{p+1}{2}, \frac{p+1}{p}) > 1$,

$$\begin{aligned}\mathbb{E}(\overline{Q}_n^M) &\leq \sum_{1 \leq \ell \leq M} (4M)^M \mathbb{E}(\varphi^{p_0})^\ell\, n^{2M\gamma - \ell p_0 \gamma + \ell} \\ &\leq c_{p,\varphi}\, n^{2M\gamma},\end{aligned}$$

where $c_{p,\varphi} = 4^M M^{M+1} \max(1, \mathbb{E}(\varphi^{p_0})^M)$. Plugging this inside (2.6), one gets

$$\mathbb{P}(A_{4,n}) \leq C_{4,n} := c_{p,\varphi} D_M\, n^{-2(1-\gamma)M},$$

and

$$\sum_{n\geq 1} n^{p-2} C_{4,n} = c_{p,\varphi} D_M \sum_{n\geq 1} n^{p-2-2(1-\gamma)M},$$

which is finite since $M \geq \frac{p}{2(1-\gamma)}$. □

Remark 2.3. As we have seen in this proof, the assumption (2.1) in Theorem 2.2 implies that there exists a constant $C := \mathbb{E}|\varphi|^p$ such that, for all $n \geq 1$

$$\mathbb{E}(|\varphi_n|^p \mid \mathcal{B}_{n-1}) \leq C \ . \tag{2.7}$$

However, the conclusion of Theorem 2.2 is no more true if we replace assumption (2.1) by (2.7). Here is a counterexample. Choose φ_n to be symmetric independant random variables such that, for $3^{i-1} < n \leq 3^i$, φ_n takes values in the set $\{-3^i, 0, 3^i\}$ and $\mathbb{P}(\varphi_n = \pm 3^i) = 3^{-pi}$. For these variables, the conclusion of Theorem 2.2 does not hold. This is essentially due to the fact that the series $\sum_{n\geq 1} n^{p-2}\mathbb{P}(\exists k \leq n \mid \ |\varphi_k| \geq n)$ diverge (the details are left to the reader since we will not use this example).

When the martingale difference is uniformly bounded, one has a much better large deviation estimate than (2.2) due to Azuma in [1].

Fact 2.4. (Azuma) *Let $(\varphi_n)_{n\geq 1}$ be a martingale difference and $S_n := \varphi_1 + \cdots + \varphi_n$ the corresponding martingale. If $|\varphi_n| \leq a < \infty$ for all $n \geq 1$, then one has for all $n \geq 1$, $\varepsilon > 0$,*

$$\mathbb{P}(S_n \geq n\varepsilon) \leq e^{-\frac{n\varepsilon^2}{2a^2}} \ . \tag{2.8}$$

Proof. We recall Azuma's proof since it is very short. Assume $a = 1$. Using the convexity of the exponential function, one bounds, for all x in $[-1, 1]$, $e^{\varepsilon x} \leq \cosh(\varepsilon) + x\sinh(\varepsilon) \leq e^{\frac{\varepsilon^2}{2}} + x\sinh(\varepsilon)$. Hence, for all $k \geq 1$, one has $\mathbb{E}(e^{\varepsilon\varphi_k} \mid \mathcal{B}_{k-1}) \leq e^{\frac{\varepsilon^2}{2}}$, and, by Chebyshev's inequality,

$$\mathbb{P}(S_n \geq n\varepsilon) \leq e^{-n\varepsilon^2}\,\mathbb{E}(e^{\varepsilon S_n}) \leq e^{-n\varepsilon^2}(e^{\frac{\varepsilon^2}{2}})^n = e^{-\frac{n\varepsilon^2}{2}}.$$

□

2.2. **Central limit theorem for martingales.**

In this section, we briefly recall the martingale central limit theorem, which is due to Brown.

Let $(\Omega, \mathcal{B}, \mathbb{P})$ be a probability space, $(p_n)_{n\geq 1}$ be a sequence of positive integers and, for $n \geq 1$, let

$$\mathcal{B}_{n,0} \subset \cdots \subset \mathcal{B}_{n,p_n}$$

be sub-σ-algebras of $\mathcal{B}$.

Let E be a finite dimensional normed real vector space. We want to define the gaussian laws N_Φ on E. Such a law is completely determined by its *covariance 2-tensor* Φ. If we fix a euclidean structure on E this *covariance 2-tensor* is nothing but the *covariance matrix* of N_Φ. Here are the precise definitions.

We denote by S^2E the space of symmetric 2-tensors of E. Equivalently, S^2E is the space of quadratic forms on the dual space E^*. The *linear span* of a symmetric 2-tensor Φ is the smallest vector subspace $E_\Phi \subset E$ such that Φ belongs to S^2E_Φ. A 2-tensor $\Psi \in \mathrm{S}^2E$ is *non-negative* (which we write $\Psi \geq 0$) if it is non-negative as a quadratic form on the dual space E^*. For every v in E we set $v^2 := v \otimes v \in \mathrm{S}^2E$, and we denote by

$$B_\Phi := \{v \in E_\Phi \mid \Phi - v^2 \text{ is non-negative}\}$$

the *unit ball* of Φ. For any non-negative symmetric 2-tensor $\Phi \in \mathrm{S}^2E$, we let N_Φ be the centered gaussian law on E with covariance 2-tensor Φ, i.e. such that,

$$\Phi = \textstyle\int_E v^2 \, \mathrm{d}N_\Phi(v).$$

For instance, N_Φ is a Dirac mass at 0 if and only if $\Phi = 0$ if and only if $E_\Phi = \{0\}$.

The following theorem is due to Brown in [17] (see also [32]).

Fact 2.5. (Brown Martingale central limit theorem) *For $1 \leq k \leq p_n$, let $\varphi_{n,k} : \Omega \to E$ be square-integrable random variables such that*

$$\mathbb{E}(\varphi_{n,k} \mid \mathcal{B}_{n,k-1}) = 0. \tag{2.9}$$

We assume that the S^2E-valued random variables

$$W_n := \sum_{1 \leq k \leq p_n} \mathbb{E}(\varphi_{n,k}^2 \mid \mathcal{B}_{n,k-1}) \quad \textit{converge to } \Phi \textit{ in probability,} \tag{2.10}$$

and that, for all $\varepsilon > 0$,

$$W_{\varepsilon,n} := \sum_{1 \leq k \leq p_n} \mathbb{E}(\varphi_{n,k}^2 \mathbf{1}_{\{\|\varphi_{n,k}\| \geq \varepsilon\}} \mid \mathcal{B}_{n,k-1}) \xrightarrow[n\to\infty]{} 0 \textit{ in probability.} \tag{2.11}$$

Then the sequence $S_n := \sum_{1 \leq k \leq p_n} \varphi_{n,k}$ converges in law toward N_Φ.

Under the same assumptions, the sequence S_n also satisfies a law of the iterated logarithm, i.e. almost surely, the set of cluster points of the sequence $\frac{S_n}{\sqrt{2\Phi\, n \log\log n}}$ is equal to the unit ball B_Φ (indeed the sequence S_n satisfies an invariance principle see [32, Chap. 4]).

The assumption (2.11) is called Lindeberg's condition.

We recall that a sequence X_n of random variables converges to X_∞ in probability, if, for all $\varepsilon > 0$, $\mathbb{P}(|X_n - X_\infty| \geq \varepsilon) \xrightarrow[n\to\infty]{} 0$.

3. Limit theorems for cocycles

In this chapter we state various limit theorems for cocycles and we explain how to deduce them from the limit theorems for martingales that we discussed in Chapter 2.

3.1. **Complete convergence for functions.**

In this section, we prove a large deviations estimate in the law of large numbers for functions over Markov-Feller chains.

Let X be a compact metrizable space and $\mathcal{C}^0(X)$ be the Banach space of continuous functions on X. Let $P : \mathcal{C}^0(X) \to \mathcal{C}^0(X)$ be a Markov-Feller operator i.e. a bounded operator such that $\|P\| \leq 1$, $P1 = 1$ and such that $Pf \geq 0$ for all functions $f \geq 0$. Such a Markov-Feller operator can be seen alternatively as a weak-$*$ continuous map $x \mapsto P_x$ from X to the set of probability measures on X, where P_x is defined by $P_x(f) = (Pf)(x)$ for all f in $\mathcal{C}^0(X)$. We denote by $\underline{X}$ the compact set $\underline{X} = X^{\mathbb{N}}$ of infinite sequences $\underline{x} = (x_0, x_1, x_2, \ldots)$. For x in X, we denote by $\mathbb{P}_x$ the Markov probability measure on $\underline{X}$ i.e. the law of the trajectories of the Markov chain starting from x associated to P.

Given a continuous function φ on X, we define its *upper average* by

$$\ell^+_\varphi = \sup_\nu \textstyle\int_G \varphi(x)\, \mathrm{d}\nu(x)$$

and *lower average* by

$$\ell^-_\varphi := \inf_\nu \textstyle\int_G \varphi(x)\, \mathrm{d}\nu(x)$$

where the supremum and the infimum are taken over all the P-invariant probability measures ν on X. We say φ *has unique average* if $\ell^+_\varphi = \ell^-_\varphi$.

According to the Breiman law of large numbers in [16] (see also [10]), for such a φ, for any x in X, for $\mathbb{P}_x$-almost every $\underline{x}$ in $\underline{X}$, the sequence $\frac{1}{n}\sum_{k=1}^n \varphi(x_k)$ converges to $\ell^+_\varphi = \ell^-_\varphi$. The following proposition is a large deviations estimate for the Breiman law of large numbers.

Proposition 3.1. *Let X be a compact metrizable space, and P be a Markov-Feller operator on X. Let φ be a continuous function on X with upper average ℓ^+_φ and lower average ℓ^-_φ. Then, for all $\varepsilon > 0$, there exist constants $A > 0$, $\alpha > 0$ such that*

$$\mathbb{P}_x\left(\left\{\underline{x} \in \underline{X} \mid \tfrac{1}{n}\textstyle\sum_{k=1}^n \varphi(x_k) \notin [\ell^-_\varphi - \varepsilon, \ell^+_\varphi + \varepsilon]\right\}\right) \;\leq\; A\, e^{-\alpha n}, \tag{3.1}$$

for all $n \geq 1$ and all x in X.

Note that $\ell^-_\varphi = \ell^+_\varphi$ as soon as P is uniquely ergodic, i.e. as soon as there exists only one P-invariant Borel probability measure ν on X.

Proof. We assume $\|\varphi\|_\infty = \frac{1}{2}$. We introduce, for $1 \leq \ell \leq n$, the bounded functions Ψ_n and $\Psi_{\ell,n}$ on $\underline{X}$ given, for $\underline{x}$ in $\underline{X}$, by

$$\Psi_n(\underline{x}) = \varphi(x_n) \text{ and } \Psi_{\ell,n}(\underline{x}) = (P^\ell\varphi)(x_{n-\ell}).$$

so that, for x in X,

$$\Psi_{\ell,n} = \mathbb{E}_x(\Psi_n \mid \mathcal{X}_{n-\ell}) \quad \mathbb{P}_x\text{-a.s.},$$

where $\mathcal{X}_n$ is the σ-algebra on $\underline{X}$ spanned by the functions $\underline{x} \mapsto x_k$ with $k \leq n$. On one hand, one has the uniform convergence

$$\max(\ell_\varphi^+, \tfrac{1}{m}\textstyle\sum_{j=1}^m P^j\varphi) \xrightarrow[m\to\infty]{} \ell_\varphi^+ \tag{3.2}$$

in $\mathcal{C}^0(X)$. Hence we can fix m such that, for all $x \in X$,

$$\tfrac{1}{m}\textstyle\sum_{j=1}^m P^j\varphi(x) \leq \ell_\varphi^+ + \frac{\varepsilon}{4}.$$

Then, for all $n \geq 1$ and $\underline{x} \in \underline{X}$, one has

$$\tfrac{1}{n\,m}\textstyle\sum_{k=m+1}^{m+n}\sum_{j=1}^m \Psi_{j,k+j}(\underline{x}) \leq \ell_\varphi^+ + \frac{\varepsilon}{4}.$$

In particular, if $n \geq n_0 := \frac{4m}{\varepsilon}$, one also has

$$\tfrac{1}{n\,m}\textstyle\sum_{k=m+1}^{m+n}\sum_{j=1}^m \Psi_{j,k}(\underline{x}) \leq \ell_\varphi^+ + \frac{\varepsilon}{2}. \tag{3.3}$$

On the other hand, for all $1 \leq j \leq m$, $x \in X$, by Azuma's bound (2.8) and the equalities, for $k \geq j$, $\mathbb{E}_x(\Psi_{j-1,k} - \Psi_{j,k} \mid \mathcal{X}_{k-j}) = 0$, one has

$$\mathbb{P}_x(\{\underline{x} \in \underline{X} \mid |\tfrac{1}{n}\textstyle\sum_{k=m+1}^{m+n}(\Psi_{j-1,k}(\underline{x}) - \Psi_{j,k}(\underline{x}))| \geq \tfrac{\varepsilon}{4m}\}) \leq e^{-\frac{n\varepsilon^2}{32m^2}}.$$

Adding these bounds, one gets, for all $1 \leq j \leq m$, $x \in X$,

$$\mathbb{P}_x(\{\underline{x} \in \underline{X} \mid |\tfrac{1}{n}\textstyle\sum_{k=m+1}^{m+n}(\Psi_k(\underline{x}) - \Psi_{j,k}(\underline{x}))| \geq \tfrac{\varepsilon}{4}\}) \leq me^{-\frac{n\varepsilon^2}{32m^2}},$$

and hence

$$\mathbb{P}_x(\{\underline{x} \in \underline{X} \mid |\tfrac{1}{n}\textstyle\sum_{k=m+1}^{m+n}(\Psi_k(\underline{x}) - \tfrac{1}{m}\sum_{j=1}^m \Psi_{j,k}(\underline{x}))| \geq \tfrac{\varepsilon}{4}\}) \leq m^2\, e^{-\frac{n\varepsilon^2}{32m^2}}.$$

Combining this formula with (3.3), one gets the desired bound,

$$\mathbb{P}_x(\{\underline{x} \in \underline{X} \mid \tfrac{1}{n}\textstyle\sum_{k=1}^n \Psi_k(\underline{x}) \geq \ell_\varphi^+ + \varepsilon\}) \leq m^2\, e^{-\frac{n\varepsilon^2}{32m^2}},$$

for all $n \geq n_0$ and $x \in X$. □

3.2. **Complete convergence for cocycles.**

In this section, we prove the complete convergence in the law of large numbers for cocycles over G-spaces.

Let G be a second countable locally compact group acting continuously on a compact second countable topological space X. Let μ be a Borel probability measure on G.

We denote by $(B, \mathcal{B}, \beta)$ the associated one-sided Bernoulli space i.e. $B = G^{\mathbb{N}^*}$ is the set of sequences $b = (b_1, \ldots, b_n, \ldots)$ with b_n in G, $\mathcal{B}$ is the product σ-algebra of the Borel σ-algebras of G, and β is the product measure $\mu^{\otimes \mathbb{N}^*}$. For $n \geq 1$, we denote by $\mathcal{B}_n$ the σ-algebra spanned by the n first coordinates $b_1, \ldots, b_n$.

We will apply the results of Section 3.1 to the *averaging operator* i.e. the Markov-Feller operator $P = P_\mu : \mathcal{C}^0(X) \to \mathcal{C}^0(X)$ whose transition probabilities are given by $P_x = \mu * \delta_x$ for all x in X. For every x in X, the Markov measure $\mathbb{P}_x$ is the image of β by the map

$$B \to \underline{X}\,;\, b \mapsto (x, b_1 x, b_2 b_1 x, b_3 b_2 b_1 x, \ldots).$$

We denote by μ^{*n} the n^{th}-convolution power $\mu * \cdots * \mu$.

Let E be a finite dimensional normed real vector space and σ a continuous function $\sigma : G \times X \to E$. This function σ is said to be a *cocycle* if one has

$$\sigma(gg', x) = \sigma(g, g'x) + \sigma(g', x) \quad \text{for any } g, g' \in G,\ x \in X. \tag{3.4}$$

We introduce the *sup-norm* function $\sigma_{\sup}$. It is given, for g in G, by

$$\sigma_{\sup}(g) = \sup_{x \in X} \|\sigma(g, x)\|\,. \tag{3.5}$$

We assume that this function $\sigma_{\sup}$ is integrable

$$\int_G \sigma_{\sup}(g)\, \mathrm{d}\mu(g) < \infty\,. \tag{3.6}$$

Recall a Borel probability measure ν on X is said to be μ-stationary if $\mu * \nu = \nu$, that is, if it is P_μ-invariant. When $E = \mathbb{R}$, we define the *upper average* of σ by

$$\sigma_\mu^+ = \sup_\nu \int_{G \times X} \sigma(g, x)\, \mathrm{d}\mu(g) \mathrm{d}\nu(x),$$

and the *lower average*

$$\sigma_\mu^- = \inf_\nu \int_{G \times X} \sigma(g, x)\, \mathrm{d}\mu(g) \mathrm{d}\nu(x),$$

where the supremum and the infimum are taken over all the μ-stationary probability measures ν on X. We say that σ *has unique average* if the averages do not depend on the choice of the μ-stationary probability measure ν i.e. if $\sigma_\mu^+ = \sigma_\mu^-$. In this case, these functions satisfy also a law of large numbers, i.e. under assumption (3.6) if σ has

unique average, for any x in X, for β-almost every b in B, the sequence $\sum_{k=1}^{n} \frac{\sigma(b_k, b_{k-1} \cdots b_1 x)}{n}$ converges to σ_μ (see [10, Chap. 2]).

The following proposition 3.2 is an analog of Baum-Katz theorem for these functions. For $p = 2$, it says that, when $\sigma_{\sup}$ is square integrable, this sequence converges completely.

Proposition 3.2. *Let G be a locally compact group, X a compact metrizable G-space, μ a Borel probability measure on G and $p > 1$. Let $\sigma : G \times X \to \mathbb{R}$ be a continuous function such that $\sigma_{\sup}$ is L^p-integrable. Let σ_μ^+ and σ_μ^- be its upper and lower average. Then, for any $\varepsilon > 0$, there exist constants D_n such that,*

$$\textstyle\sum_{n \geq 1} n^{p-2} D_n < \infty \ ,$$

and, for $n \geq 1$, $x \in X$,

$$\beta(\{b \in B \mid \textstyle\sum_{k=1}^{n} \frac{\sigma(b_k, b_{k-1} \cdots b_1 x)}{n} \notin [\sigma_\mu^- - \varepsilon, \sigma_\mu^+ + \varepsilon]\}) \leq D_n$$

In particular, when σ is a cocycle, one has, for $n \geq 1$, $x \in X$,

$$\mu^{*n}(\{g \in G \mid \tfrac{\sigma(g,x)}{n} \notin [\sigma_\mu^- - \varepsilon, \sigma_\mu^+ + \varepsilon]\}) \leq D_n \ . \tag{3.7}$$

The fact that the constants D_n do not depend on x will be important for our applications.

Proof. According to Proposition 3.1, the conclusion of Proposition 3.2 is true when the function σ does not depend on the variable g. Hence it is enough to prove Proposition 3.2 for the continuous function σ' on $G \times X$ given, for g in G and x in X, by

$$\sigma'(g, x) = \sigma(g, x) - \textstyle\int_G \sigma(g, x) \, \mathrm{d}\mu(g).$$

By construction, the sequence of functions φ_n on B given, for b in B, by

$$\varphi_n(b) = \sigma'(b_n, b_{n-1} \cdots b_1 x)$$

is a martingale difference. Hence our claim follows from Theorem 2.2 since the functions φ_n satisfy the domination (2.1): for $n \geq 1$, $t > 0$,

$$\mathbb{E}(\mathbf{1}_{\{|\varphi_n| > t\}} \mid \mathcal{B}_{n-1}) \leq \mu(\{g \in G \mid \sigma_{\sup}(g) + M > t\}) \ .$$

where M is the constant $M := \int_G \sigma_{\sup}(g) \, \mathrm{d}\mu(g)$. □

3.3. **Central limit theorem for centerable cocycles.**

In this section we explain how to deduce the central limit theorem for centerable cocycles from the central limit theorem for martingales.

Let $\sigma : G \times X \to E$ be a continuous cocycle. When the function $\sigma_{\sup}$ is μ-integrable, one defines the *drift* or *expected increase* of σ: it is the continuous function $X \to E; x \mapsto \int_G \sigma(g,x)\,\mathrm{d}\mu(g)$. One says that σ has *constant drift* if the drift is a constant function:

$$\int_G \sigma(g,x)\,\mathrm{d}\mu(g) = \sigma_\mu. \tag{3.8}$$

One says that σ is *centered* if the drift is a null function.

A continuous cocycle $\sigma : G \times X \to E$ is said to be *centerable* if it is the sum

$$\sigma(g,x) = \sigma_0(g,x) + \psi(x) - \psi(gx) \tag{3.9}$$

of a cocycle $\sigma_0(g,x)$ with constant drift σ_μ and of a coboundary $\psi(x) - \psi(gx)$ given by a continuous function $\psi \in \mathcal{C}^0(X)$. A centerable cocycle always has a unique average: for any μ-stationary probability ν on X, one has

$$\int_{G\times X} \sigma(g,x)\,\mathrm{d}\mu(g)\,\mathrm{d}\nu(x) = \sigma_\mu.$$

Here is a trick to reduce the study of a cocycle with constant drift σ_μ to one which is centered. Replace G by $G' := G \times \mathbb{Z}$ where $\mathbb{Z}$ acts trivially on X, replace μ by $\mu' := \mu \otimes \delta_1$, so that any μ-stationary probability measure on X is also μ'-stationary, and replace σ by the cocycle

$$\sigma' : G' \times X \to E \text{ given by } \quad \sigma'((g,n),x) = \sigma(g,x) - n\sigma_\mu. \tag{3.10}$$

A centerable cocycle σ is said to *have unique covariance* Φ_μ if

$$\Phi_\mu := \int_{G\times X} (\sigma_0(g,x) - \sigma_\mu)^2\,\mathrm{d}\mu(g)\,\mathrm{d}\nu(x) \text{ does not depend on the choice of the } \mu\text{-stationary probability measure } \nu, \tag{3.11}$$

where σ_0 is as in (3.9). This covariance 2-tensor $\Phi_\mu \in \mathrm{S}^2 E$ is non-negative.

Remark 3.3. This assumption does not depend on the choice of σ_0. More precisely, if σ_0 and σ_1 are cohomologous centered cocycles, for any μ-stationary Borel probability measure ν on X, one has

$$\int_{G\times X} \sigma_0(g,x)^2\,\mathrm{d}\mu(g)\,\mathrm{d}\nu(x) = \int_{G\times X} \sigma_1(g,x)^2\,\mathrm{d}\mu(g)\,\mathrm{d}\nu(x). \tag{3.12}$$

Indeed, since σ_0 and σ_1 are centered and cohomologous, we may write, for any g, x, $\sigma_1(g,x) = \sigma_0(g,x) + \psi(x) - \psi(gx)$ where ψ is a continuous function on X and $P_\mu\psi = \psi$. Now, the difference between the two sides of (3.12) reads as

$$2\int_{G\times X} \sigma_0(g,x)\psi(gx)\,\mathrm{d}\mu(g)\,\mathrm{d}\nu(x). \tag{3.13}$$

By ergodic decomposition, to prove this is 0, one can assume ν is μ-ergodic. In this case, since $P_\mu\psi = \psi$, ψ is constant ν-almost everywhere

and (3.13) is proportional to $\int_{G\times X} \sigma_0(g,x)\,\mathrm{d}\mu(g)\,\mathrm{d}\nu(x)$, which is 0 by assumption.

Theorem 3.4. (Central limit theorem for centerable cocycles) *Let G be a locally compact group, X a compact metrizable G-space, E a finite dimensional real vector space, and μ a Borel probability measure on G. Let $\sigma : G \times X \to E$ be a continuous cocycle such that $\int_G \sigma_{\sup}(g)^2\,\mathrm{d}\mu(g) < \infty$. Assume that σ is centerable with average σ_μ and has a unique covariance Φ_μ i.e. σ satisfies* (3.9) *and* (3.11). *Let N_μ be the Gaussian law on E whose covariance 2-tensor is Φ_μ.*

Then, for any bounded continuous function ψ on E, uniformly for x in X, one has

$$\int_G \psi\left(\tfrac{\sigma(g,x)-n\sigma_\mu}{\sqrt{n}}\right)\,\mathrm{d}\mu^{*n}(g) \xrightarrow[n\to\infty]{} \int_E \psi(v)\,\mathrm{d}N_\mu(v)\ . \tag{3.14}$$

Note that Hypothesis (3.11) is automatically satisfied when there exists a unique μ-stationary Borel probability measure ν on X.

Remarks 3.5. When $E = \mathbb{R}^d$, the covariance 2-tensor Φ_μ is nothing but the *covariance matrix* of the random variable σ_0 on $(G\times X, \mu\otimes\nu)$.

The conclusion in Theorem 3.4 is not correct if one does not assume the cocycle σ to be centerable.

Proof. We will deduce Theorem 3.4 from the central limit theorem 2.5 for martingales.

As in the previous sections, let $(B,\mathcal{B},\beta)$ be the Bernoulli space with alphabet (G,μ). We want to prove that, for any sequence x_n on X, the laws of the random variables S_n on B given, for b in B, by

$$S_n(b) := \tfrac{1}{\sqrt{n}}(\sigma(b_n\cdots b_1, x_n) - n\sigma_\mu)$$

converge to N_μ.

Since the cocycle σ is centerable, one can write σ as the sum of two cocycles $\sigma = \sigma_0 + \sigma_1$ where σ_0 has constant drift and where σ_1 is a coboundary. In particular the cocycle σ_1 is uniformly bounded and does not play any role in the limit (3.14). Hence we can assume $\sigma = \sigma_0$. Using the trick (3.10), we can assume that $\sigma_\mu = 0$ i.e. that σ is a centered cocycle.

We want to apply the martingale central limit theorem 2.5 to the sub-σ-algebras $\mathcal{B}_{n,k} = \mathcal{B}_k$ spanned by $b_1,\dots,b_k$ and to the triangular array of random variables $\varphi_{n,k}$ on B given by, for b in B,

$$\varphi_{n,k}(b) = \tfrac{1}{\sqrt{n}}\sigma(b_k, b_{k-1}\cdots b_1 x_n), \quad \text{for } 1\le k\le n\ .$$

Since, by the cocycle property (3.4), one has

$$S_n = \textstyle\sum_{1\le k\le n}\varphi_{n,k}\ ,$$

we just have to check that the three assumptions of Theorem 2.5 are satisfied with $\Phi = \Phi_\mu$. We keep the notations W_n and $W_{\varepsilon,n}$ of this theorem.

First, since the function $\sigma_{\sup}$ is square integrable, the functions $\varphi_{n,k}$ belong to $\mathrm{L}^2(B,\beta)$, and, by Equation (3.8), the assumption (2.9) is satified: for β-almost all b in B,

$$\mathbb{E}(\varphi_{n,k} \mid \mathcal{B}_{k-1}) = \int_G \sigma(g, b_{k-1}\cdots b_1 x_n)\,\mathrm{d}\mu(g) = 0\ .$$

Second, we introduce the continuous function on X,

$$x \mapsto M(x) = \int_G \sigma(g,x)^2\,\mathrm{d}\mu(g)\ .$$

and we compute, for β-almost all b in B,

$$W_n(b) = \tfrac{1}{n}\textstyle\sum_{1\le k\le n} M(b_{k-1}...b_1 x_n).$$

According to Proposition 3.1, since σ has a unique covariance Φ_μ, the sequence W_n converges to Φ_μ in probability, i.e. the assumption (2.10) is satisfied.

Third, we introduce, for $\lambda > 0$, the continuous function on X

$$x \mapsto M_\lambda(x) = \int_G \sigma(g,x)^2 \mathbf{1}_{\{\|\sigma(g,x)\|\ge\lambda\}}\,\mathrm{d}\mu(g)\ .$$

and the integral

$$I_\lambda := \int_G \sigma_{\sup}^2(g)\mathbf{1}_{\{\sigma_{\sup}(g)\ge\lambda\}}\,\mathrm{d}\mu(g)\ ,$$

we notice that

$$M_\lambda(x) \le I_\lambda \xrightarrow[\lambda\to\infty]{} 0\,,$$

and we compute, for $\varepsilon > 0$ and β-almost all b in B,

$$W_{\varepsilon,n}(b) = \tfrac{1}{n}\textstyle\sum_{1\le k\le n} M_{\varepsilon\sqrt{n}}(b_{k-1}...b_1 x_n) \le I_{\varepsilon\sqrt{n}} \xrightarrow[n\to\infty]{} 0\ .$$

In particular the sequence $W_{\varepsilon,n}$ converges to 0 in probability, i.e. Lindeberg's condition (2.11) is satisfied.

Hence, by Fact 2.5, the laws of S_n converge to N_μ. □

4. Limit theorems for linear groups

In this chapter, we prove the central limit theorem for linear groups (Theorem 1.1). Our main task will be to prove that the norm cocycle (1.5) is centerable

4.1. **Complete convergence for linear groups.**

In this section we prove the complete convergence in the Furstenberg law of large numbers.

Let $\mathbb{K}$ be a local field. The reader who is not familiar with local fields may assume $\mathbb{K} = \mathbb{R}$. In general, a local field is a non-discrete locally compact field. It is a classical fact that such a field is a finite extension of either
(i) the field $\mathbb{R}$ of real numbers (in this case, one has $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), or
(ii) the field $\mathbb{Q}_p$ of p-adic numbers, for some prime number p, or
(iii) the field $\mathbb{F}_p((t))$ of Laurent series with coefficients in the finite field $\mathbb{F}_p$ of cardinality p, for some prime number p.

Let V be a finite dimensional $\mathbb{K}$-vector space. We fix a basis $e_1, \dots, e_d$ of V and the following norm on V. For $v = \sum v_i e_i \in V$ we set $\|v\| = (\sum |v_i|^2)^{\frac{1}{2}}$ when $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and $\|v\| = \max(|v_i|)$ in the other cases. We denote by $e_1^*, \dots, e_d^*$ the dual basis of V^* and we use the same symbol $\|.\|$ for the norms induced on the dual space V^*, on the space $\mathrm{End}(V)$ of endomorphisms of V, or on the exterior product $\wedge^2 V$, etc... We equip the projective space $\mathbb{P}(V)$ with the distance d given, by

$$d(x, x') = \tfrac{\|v \wedge v'\|}{\|v\| \, \|v'\|} \text{ for } x = \mathbb{K}v,\ x' = \mathbb{K}v' \text{ in } \mathbb{P}(V).$$

For g in $\mathrm{GL}(V)$, we write $N(g) := \max(\|g\|, \|g^{-1}\|)$.

Let μ be a Borel probability measure on $G := \mathrm{GL}(V)$ with finite first moment: $\int_G \log N(g) \, \mathrm{d}\mu(g) < \infty$. We denote by Γ_μ the subsemigroup of G spanned by the support of μ, and by λ_1 the first Lyapunov exponent of μ,

$$\lambda_1 := \lim_{n \to \infty} \tfrac{1}{n} \int_G \log \|g\| \, \mathrm{d}\mu^{*n}(g). \tag{4.1}$$

Let $b_1, \dots, b_n, \dots$ be random elements of G chosen independantly with law μ. The Furstenberg law of large numbers describes the behavior of the random variables $\log \|b_n \cdots b_1\|$. It is a direct consequence of the Kingman subadditive ergodic theorem (see for example [43]). It states that, for $\mu^{\otimes \mathbb{N}^*}$-almost any sequence $(b_1, \dots, b_n, \dots)$ in G, one has

$$\lim_{n \to \infty} \tfrac{1}{n} \log \|b_n \cdots b_1\| = \lambda_1. \tag{4.2}$$

The following Proposition 4.1 is an analogue of the Baum-Katz theorem for the Furstenberg law of large numbers. For $p = 2$, it says that, when the second moment of μ is finite, this sequence (4.2) converges completely.

Proposition 4.1. *Let $p > 1$ and $V = \mathbb{K}^d$. Let μ be a Borel probability measure on the group $G := \mathrm{GL}(V)$, such that the p^{th}-moment*

$\int_G (\log N(g))^p \, \mathrm{d}\mu(g)$ is finite. Then, for every $\varepsilon > 0$, there exist constants $C_n = C_n(p, \varepsilon, \mu)$ such that $\sum_{n \geq 1} n^{p-2} C_n < \infty$ and

$$\mu^{*n}(\{g \in G \text{ such that } |\log \|g\| - n\lambda_1| \geq \varepsilon n\}) \leq C_n. \tag{4.3}$$

Moreover, if Γ_μ acts irreducibly on V, for any v in $V \smallsetminus \{0\}$, one has

$$\mu^{*n}(\{g \in G \text{ such that } |\log \tfrac{\|gv\|}{\|v\|} - n\lambda_1| \geq \varepsilon n\}) \leq C_n, \tag{4.4}$$

Proof. We first prove the claim (4.3). We fix $\varepsilon > 0$. We will apply Proposition 3.2 to the group $G = \mathrm{GL}(V)$ acting on the projective space $X = \mathbb{P}(V)$ and to the norm cocycle

$$\sigma : G \times X \to \mathbb{R} \; ; \; (g, \mathbb{K}v) \mapsto \log \tfrac{\|gv\|}{\|v\|}$$

for which the function $\sigma_{\sup}$ is L^p-integrable. According to Furstenberg-Kifer and Hennion theorem in [22, Th. 3.9 & 3.10] and [33, Th. 1& Cor. 2] (see also [10, Ch. 3]), the Lyapunov exponent λ_1 is the upper average of σ i.e.

$$\lambda_1 = \sup_\nu \textstyle\int_{G \times X} \sigma(g, x) \, \mathrm{d}\mu(g) \, \mathrm{d}\nu(x),$$

and there exists a unique Γ_μ-invariant vector subspace $V' \subset V$ such that, on one hand, the first Lyapunov exponent λ_1' of the image μ' of μ in $\mathrm{GL}(V')$ is strictly smaller than λ_1, and, on the other hand, the image μ'' of μ in $\mathrm{GL}(V'')$ with $V'' = V/V'$ has exponent λ_1 and the cocycle $\sigma'' : \mathrm{GL}(V'') \times \mathbb{P}(V'') \to \mathbb{R} \; ; \; (g, \mathbb{K}v) \mapsto \log \frac{\|gv\|}{\|v\|}$ has unique average λ_1.

Since λ_1 is the upper average of σ, by Proposition 3.2, there exist constants $C_n = C_n(p, \varepsilon, \mu)$ such that $\sum_{n \geq 1} n^{p-2} C_n < \infty$ and, for all v in $V \smallsetminus \{0\}$ and $n \geq 1$,

$$\mu^{*n}(\{g \in G \mid \log \tfrac{\|gv\|}{\|v\|} - n\lambda_1 \geq \varepsilon n\}) \leq C_n. \tag{4.5}$$

Since λ_1 is the unique average of σ'', using again Proposition 3.2, one can choose C_n such that, for all v'' in $V'' \smallsetminus \{0\}$ and $n \geq 1$,

$$\mu^{*n}(\{g \in G \mid \log \tfrac{\|gv''\|}{\|v''\|} - n\lambda_1 \notin [-\varepsilon n, \varepsilon n]\}) \leq C_n, \tag{4.6}$$

where, as usual, the norm in the quotient space V'' is defined by the equality $\|v''\| = \inf\{\|v\| \mid v \in v'' + V'\}$.

The claim (4.3), with a different constant C_n, follows from a combination of the claim (4.5) applied to a basis $v_1, \ldots, v_d$ of V and from the claim (4.6) applied to a non-zero vector v'' in V''. One just has to notice that there exists a positive constant M such that one has

$$\log \tfrac{\|gv''\|}{\|v''\|} \leq \log \|g\| \leq \max_{1 \leq i \leq d} \log \tfrac{\|gv_i\|}{\|v_i\|} + M \; ,$$

for all g in $\mathrm{GL}(V)$ preserving V'.

The claim (4.4) follows from (4.6), since, when the action of Γ_μ on V is irreducible, one has $V'' = V$. □

We denote by λ_2 the second Lyapounov exponent of μ, i.e.

$$\lambda_2 := \lim_{n\to\infty} \tfrac{1}{n} \int_G \log \tfrac{\|\wedge^2 g\|}{\|g\|}\, \mathrm{d}\mu^{*n}(g). \tag{4.7}$$

Corollary 4.2. *Assume the same assumptions as in Proposition 4.1. For every $\varepsilon > 0$, there exist constants C_n such that $\sum_{n\geq 1} n^{p-2} C_n < \infty$ and*

$$\mu^{*n}(\{g \in G \text{ such that } |\log\|\wedge^2 g\| - n(\lambda_1+\lambda_2)| \geq \varepsilon n\}) \leq C_n. \tag{4.8}$$

Proof. Our statement (4.8) is nothing but (4.3) applied to $\wedge^2 V$. □

Remarks 4.3. An endomorphism g of V is said to be proximal if it admits an eigenvalue λ which has multiplicity one and if all other eigenvalues of g have modulus $< |\lambda|$. The action of Γ_μ on V is said to be *proximal* if Γ_μ contains a proximal endomorphism. The action of Γ_μ on V is said to be *strongly irreducible* if no proper finite union of vector subspaces of V is Γ_μ-invariant.

According to a result of Furstenberg (see for example [14]), when Γ_μ is unbounded, included in $\mathrm{SL}(V)$ and strongly irreducible in V, the first Lyapounov exponent is positive: $\lambda_1 > 0$.

According to a result of Guivarc'h in [27], when the action of Γ_μ is proximal and strongly irreducible, the first Lyapounov exponent is simple i.e. one has $\lambda_1 > \lambda_2$. We will use this fact in the next section.

4.2. Log-regularity in projective space.

In this section, we prove the log-regularity of the Furstenberg measure for proximal stronly irreducible representations when the second moment of μ is finite.

For any $y = \mathbb{K}f$ in $\mathbb{P}(V^*)$, we set $y^\perp \subset \mathbb{P}(V)$ for the orthogonal projective hyperplane: $y^\perp = \mathbb{P}(\mathrm{Ker} f)$. For $x = \mathbb{K}v$ in $\mathbb{P}(V)$ and $y = \mathbb{K}f$ in $\mathbb{P}(V^*)$, we set

$$\delta(x, y) = \tfrac{|f(v)|}{\|f\|\|v\|}.$$

This quantity is also equal to the distance $\delta(x, y) = d(x, y^\perp)$ in $\mathbb{P}(V)$ and to the distance $d(y, x^\perp)$ in $\mathbb{P}(V^*)$.

Remark 4.4. Let μ be a Borel probability measure on $\mathrm{GL}(V)$ such that Γ_μ is proximal and strongly irreducible on V. Then, due to a result of Furstenberg, μ admits a unique μ-stationary Borel probability measure ν on $\mathbb{P}(V)$. For β-almost any b in B, the sequence of Borel probability measures $(b_1 \cdots b_n)_*\nu$ converges to a Dirac measure (see [14, III.4] in the real case and [10, Chap. 3] in the general case).

Proposition 4.5. *Let $p > 1$ and $V = \mathbb{K}^d$. Let μ be a Borel probability measure on $G = \mathrm{GL}(V)$ whose p^{th}-moment is finite. Assume that Γ_μ is proximal and strongly irreducible on V. Let ν be the unique μ-stationary Borel probability measure on $X = \mathbb{P}(V)$. Then, for all y in $\mathbb{P}(V^*)$,*

$$\int_X |\log \delta(x,y)|^{p-1} \, \mathrm{d}\nu(x) \quad \textit{is finite,} \tag{4.9}$$

and is a continuous function of y.

Remarks 4.6. By a theorem of Guivarc'h in [28], when μ is assumed to have an exponential moment, the stationary measure ν is much more regular: its Hausdorff dimension is finite, i.e. there exists $t > 0$ such that

$$\sup_{y \in \mathbb{P}(V^*)} \int_X \delta(x,y)^{-t} \, \mathrm{d}\nu(x) < \infty. \tag{4.10}$$

The following proof of Proposition 4.5 is similar to our proof in [10] of Guivarc'h theorem, which is inspired by [15].

Note that the integral (4.9) may be infinite when the action of Γ_μ is assumed to be "irreducible" instead of "strongly irreducible" (see Example 4.15).

Let K be the group of isometries of $(V, \|.\|)$ and A^+ be the semigroup

$$A^+ := \{\mathrm{diag}(a_1, \dots, a_d) \mid |a_1| \geq \cdots \geq |a_d|\}.$$

For every element g in $\mathrm{GL}(V)$, we choose a decomposition

$$g = k_g a_g \ell_g \ \text{ with } k_g, \ell_g \text{ in } K \text{ and } a_g \text{ in } A^+.$$

We denote by $x_g^M \in \mathbb{P}(V)$ the *density point* of g and by $y_g^m \in \mathbb{P}(V^*)$ the *density point* of ${}^t g$, that is

$$x_g^M := \mathbb{K} k_g e_1 \ \text{ and } \ y_g^m := \mathbb{K} \, {}^t\ell_g e_1^*.$$

We denote by $\gamma_1(g)$ the *first gap* of g, that is, $\gamma_1(g) := \frac{\|\wedge^2 g\|}{\|g\|^2}$.

Lemma 4.7. *For every g in $\mathrm{GL}(V)$, $x = \mathbb{K}v$ in $\mathbb{P}(V)$ and $y = \mathbb{K}f$ in $\mathbb{P}(V^*)$, one has*
(i) $\delta(x, y_g^m) \leq \frac{\|gv\|}{\|g\|\|v\|} \leq \delta(x, y_g^m) + \gamma_1(g)$
(ii) $\delta(x_g^M, y) \leq \frac{\|{}^t g f\|}{\|g\|\|f\|} \leq \delta(x_g^M, y) + \gamma_1(g)$
(iii) $d(gx, x_g^M)\, \delta(x, y_g^m) \leq \gamma_1(g)$.

Proof. For all these inequalities, we can assume that g belongs to A^+, i.e. $g = \mathrm{diag}(a_1, \dots, a_d)$ with $|a_1| \geq \cdots \geq |a_d|$. We write $v = v_1 + v_2$ with v_1 in $\mathbb{K}e_1$ and v_2 in the Kernel of e_1^*. One has then

$$\|g\| = |a_1| \ , \ \ \gamma_1(g) = \tfrac{|a_2|}{|a_1|}, \ \text{ and } \ \delta(x, y_g^m) = \tfrac{\|v_1\|}{\|v\|}.$$

(i) follows from $\|g\|\,\|v_1\| \leq \|gv\| \leq \|g\|\,\|v_1\| + |a_2|\,\|v_2\|$.
(ii) follows from (i) by replacing V with V^* and g with ${}^t g$.
(iii) follows from $d(gx, x_g^M)\,\delta(x, y_g^m) = \frac{\|gv_2\|}{\|gv\|}\,\frac{\|v_1\|}{\|v\|} \leq \frac{|a_2|}{|a_1|}$. □

Lemma 4.8. *Under the same assumptions as Proposition 4.5, there exist constants $c > 0$, and $C_n > 0$ with $\sum_{n\geq 1} n^{p-2} C_n < \infty$, and such that, for $n \geq 1$, x in $\mathbb{P}(V)$ and y in $\mathbb{P}(V^*)$, one has*

$$\mu^{*n}(\{g \in G \mid d(gx, x_g^M) \geq e^{-cn}\}) \leq C_n, \tag{4.11}$$

$$\mu^{*n}(\{g \in G \mid \delta(x_g^M, y) \leq e^{-cn}\}) \leq C_n, \tag{4.12}$$

$$\mu^{*n}(\{g \in G \mid \delta(gx, y) \leq e^{-cn}\}) \leq C_n. \tag{4.13}$$

Proof. We set $c = \frac{1}{2}(\lambda_1 - \lambda_2)$ where λ_1 and λ_2 are the first two Lyapunov exponents of μ (see Section 4.1). According to Guivarc'h theorem in [27], since the action of Γ_μ is proximal and strongly irreducible, one has $\lambda_1 > \lambda_2$. According to Proposition 4.1 and its Corollary 4.2, there exist constants C_n such that $\sum_{n\geq 1} n^{p-2} C_n < \infty$ and such that, for $n \geq 1$, $x = \mathbb{K}v$ in $\mathbb{P}(V)$ and $y = \mathbb{K}f$ in $\mathbb{P}(V^*)$ with $\|v\| = \|\varphi\| = 1$, there exist subsets $G_{n,x,y} \subset G$ with $\mu^{*n}(G_{n,x,y}) \geq 1 - C_n$, such that, for g in $G_{n,x,y}$, the four quantities

$$\left|\lambda_1 - \tfrac{\log\|g\|}{n}\right|, \left|\lambda_1 - \tfrac{\log\|gv\|}{n}\right|, \left|\lambda_1 - \tfrac{\log\|{}^t g\varphi\|}{n}\right|, \left|\lambda_1 - \lambda_2 - \tfrac{\log\gamma_1(g)}{n}\right|$$

are bounded by $\varepsilon\,(\lambda_1 - \lambda_2)$ with $\varepsilon = \frac{1}{8}$. We will choose n_0 large enough, and prove the bounds (4.11), (4.12) and (4.13) only for $n \geq n_0$. We have to check that, for $n \geq n_0$ and g in $G_{n,x,y}$, one has

$$d(gx, x_g^M) \leq e^{-cn}\ ,\ \ \delta(x_g^M, y) \geq e^{-cn} \ \text{ and } \ \delta(gx, y) \geq e^{-cn}.$$

We first notice that, according to Lemma 4.7.i, one has

$$\delta(x, y_g^m) \geq e^{-2\varepsilon\,(\lambda_1 - \lambda_2)n} - e^{-(1-\varepsilon)(\lambda_1 - \lambda_2)n}$$

hence, since n_0 is arbitrarily large,

$$\delta(x, y_g^m) \geq e^{-3\varepsilon(\lambda_1 - \lambda_2)n} \tag{4.14}$$

But then, using Lemma 4.7.iii one gets, for n_0 large enough,

$$d(gx, x_g^M) \leq e^{-(1-\varepsilon)(\lambda_1 - \lambda_2)n} e^{3\varepsilon\,(\lambda_1 - \lambda_2)n} = e^{-(1-4\varepsilon)(\lambda_1 - \lambda_2)n}. \tag{4.15}$$

This proves (4.11).

Applying the same argument as above to ${}^t g$ acting on $\mathbb{P}(V^*)$, the inequality (4.14) becomes

$$\delta(x_g^M, y) \geq e^{-3\varepsilon\,(\lambda_1 - \lambda_2)n}. \tag{4.16}$$

This proves (4.12).

Hence, combining (4.16) with (4.15), one gets, for n_0 large enough,

$$\begin{aligned}\delta(gx,y) &\geq \delta(x_g^M,y) - d(gx,x_g^M)\\ &\geq e^{-3\varepsilon(\lambda_1-\lambda_2)n} - e^{-(1-4\varepsilon)(\lambda_1-\lambda_2)n} \geq e^{-4\varepsilon\,(\lambda_1-\lambda_2)n}.\end{aligned}$$

This proves (4.13). □

Proof of Proposition 4.5. We choose c, C_n as in Lemma 4.8. We first check that, for $n \geq 1$ and y in $\mathbb{P}(V^*)$, one has

$$\nu(\{x \in X \mid \delta(x,y) \leq e^{-cn}\}) \leq C_n. \tag{4.17}$$

Indeed, since $\nu = \mu^{*n} * \nu$, one computes using (4.13)

$$\begin{aligned}\nu(\{x\in X \mid \delta(x,y)\leq e^{-cn}\}) &= \textstyle\int_X \mu^{*n}(\{g\in G \mid \delta(gx,y)\leq e^{-cn}\})\,\mathrm{d}\nu(x)\\ &\leq \textstyle\int_X C_n\,\mathrm{d}\nu(x) = C_n,\end{aligned}$$

Then cutting the integral (4.10) along the subsets $A_{n-1,y} \smallsetminus A_{n,y}$ where

$$A_{n,y} := \{x \in X \mid \delta(x,y) \leq e^{-cn}\}$$

one gets the upperbound

$$\begin{aligned}\textstyle\int_X |\log\delta(x,y)|^{p-1}\,\mathrm{d}\nu(x) &\leq \textstyle\sum_{n\geq 1} c^{p-1}n^{p-1}(\nu(A_{n-1,y}) - \nu(A_{n,y}))\\ &\leq c^{p-1} + c^{p-1}\textstyle\sum_{n\geq 1}((n+1)^{p-1} - n^{p-1})\,C_n\\ &\leq c^{p-1} + (p-1)\,2^p c^{p-1}\textstyle\sum_{n\geq 1} n^{p-2}C_n.\end{aligned}$$

which is finite. This proves (4.9).

It remains to check the continuity of the function on $\mathbb{P}(V^*)$

$$\psi^* : y \mapsto \textstyle\int_X |\log\delta(x,y)|^{p-1}\,\mathrm{d}\nu(x).$$

The fact that the above constants C_n do not depend on y tells us that this function ψ^* is a uniform limit of continuous functions ψ_n^* given by

$$\psi_n^* : y \mapsto \textstyle\int_X \min(|\log\delta(x,y)|, cn)^{p-1}\,\mathrm{d}\nu(x).$$

Hence the function ψ^* is continuous. □

4.3. **Solving the cohomological equation.**

In this section, we prove that the norm cocycle is centerable.

We recall that the norm cocycle σ on $X = \mathbb{P}(V)$ is the cocycle

$$\sigma : \mathrm{GL}(V) \times \mathbb{P}(V) \to \mathbb{R}\ ;\ (g, \mathbb{K}v) \mapsto \log\tfrac{\|gv\|}{\|v\|}.$$

Proposition 4.9. *Let μ be a Borel probability measure on $G = \mathrm{GL}(\mathbb{K}^d)$ whose second moment is finite. Assume that Γ_μ is proximal and strongly irreducible on $V := \mathbb{K}^d$. Then the norm cocycle σ on $\mathbb{P}(V)$ is centerable i.e. satisfies* (3.9).

Proof. Let

$$\varphi : x \mapsto \textstyle\int_G \sigma(g,x)\,\mathrm{d}\mu(g) \tag{4.18}$$

be the expected increase of the cocycle σ. We want to find a continuous function ψ on X such that

$$\varphi = \psi - P_\mu\psi + \lambda_1, \tag{4.19}$$

where $P_\mu\psi(x) = \int_G \psi(gx)\,\mathrm{d}\mu(g)$, for all x in X, and where λ_1 is the first exponent of μ on V.

Let $\check\mu$ be the image of μ by $g \mapsto g^{-1}$. We will also denote by σ the norm cocycle on $\mathbb{P}(V^*)$ i.e. the cocycle

$$\sigma : \mathrm{GL}(V) \times \mathbb{P}(V^*) \to \mathbb{R}\ ;\ (g, \mathbb{K}f) \mapsto \log \tfrac{\|f\circ g^{-1}\|}{\|f\|}.$$

Since the representation of $\Gamma_{\check\mu}$ in V^* is also proximal and strongly irreducible, there exists a unique $\check\mu$-stationary probability measure ν^* on the dual projective space $\mathbb{P}(V^*)$.

Since the second moment of μ is finite, according to Proposition 4.5, this measure ν^* is log-regular. Hence the following formula defines a continuous function ψ on X,

$$\psi(x) = \textstyle\int_G \log \delta(x,y)\,\mathrm{d}\nu^*(y), \tag{4.20}$$

where $\delta(x,y) = \frac{|f(v)|}{\|f\|\,\|v\|}$, for $x = \mathbb{R}v$ in $\mathbb{P}(V)$ and $y = \mathbb{R}f$ in $\mathbb{P}(V^*)$,

We check the equality,

$$\sigma(g,x) = \log \delta(x, g^{-1}y) - \log \delta(gx, y) + \sigma(g^{-1}, y) \tag{4.21}$$

by computing each side,

$$\log \tfrac{\|gv\|}{\|v\|} = \log \tfrac{|f(gv)|}{\|f\circ g\|\,\|v\|} - \log \tfrac{|f(gv)|}{\|f\|\,\|gv\|} + \log \tfrac{\|f\circ g\|}{\|f\|}.$$

Integrating Equation (4.21) on $G\times\mathbb{P}(V^*)$ for the measure $\mathrm{d}\mu(g)\,\mathrm{d}\nu^*(y)$ and using the $\check\mu$-stationarity of ν^*, one gets (4.19) since λ_1 is also the first exponent of $\check\mu$ in V^*. □

4.4. **Central limit theorem for linear groups.**

> The tools we have developped so far allow us to prove not only our central limit theorem 1.1 but also a multi-dimensional version of this theorem.

For $i = 1, \dots, m$, let $\mathbb{K}_i$ be a local field and V_i be a finite dimensional normed $\mathbb{K}_i$-vector space, and let μ be a Borel probability measure on the locally compact group $G := \mathrm{GL}(V_1) \times \cdots \times \mathrm{GL}(V_m)$. We assume that Γ_μ acts strongly irreducibly in each V_i. We consider the compact space $X = \mathbb{P}(V_1) \times \cdots \times \mathbb{P}(V_m)$.

We denote by $\sigma : G \times X \to \mathbb{R}^m$ the *multinorm cocycle*, that is, the continuous cocycle given, for $g = (g_1, \dots, g_m)$ in G and $x = (\mathbb{K}_1 v_1, \dots, \mathbb{K}_m v_m)$ in X, by

$$\sigma(g, x) := (\log \tfrac{\|g_1 v_1\|}{\|v_1\|}, \dots, \log \tfrac{\|g_m v_m\|}{\|v_m\|}).$$

We introduce also the function $\kappa : G \to \mathbb{R}^m$ given, for g in G, by

$$\kappa(g) := (\log \|g_1\|, \dots, \log \|g_m\|)$$

and the function $\ell : G \to \mathbb{R}^m$ given by

$$\ell(g) := \lim_{n\to\infty} \tfrac{1}{n}\, \kappa(g^n),$$

so that, the i^{th} coefficient of $\ell(g)$ is the logarithm of the spectral radius of g_i. For g in G, we set $N(g) = \sum_{i=1}^m N(g_i)$.

Remark 4.10. Let μ be a Borel probability measure on the group $\mathrm{GL}(V_1) \times \ldots \times \mathrm{GL}(V_m)$ such that, for any $1 \leq i \leq m$, Γ_μ is proximal and strongly irreducible in V_i. By Remark 4.4, μ admits a unique μ-stationary Borel probability measure ν_i on $\mathbb{P}(V_i)$ and, for β-almost any b in B, $(b_1 \cdots b_n)_* \nu_i$ converges towards a Dirac mass $\delta_{\xi_i(b)}$ as $n \to \infty$. One easily shows that this implies that the image ν of β by the map

$$B \to X \ ; \ b \mapsto (\xi_1(b), \dots, \xi_m(b))$$

is the unique μ-stationary Borel probability measure on X (see for example [10, Chap. 1]).

Here is the mutidimensional version of Theorem 1.1.

Theorem 4.11. *Let μ be a Borel probability measure on the group $G := \mathrm{GL}(V_1) \times \ldots \times \mathrm{GL}(V_m)$ such that Γ_μ acts strongly irreducibly on each V_i, and such that $\int_G (\log N(g))^2 \, \mathrm{d}\mu(g) < \infty$.*
a) There exist an element λ in $\mathbb{R}^m$, and a gaussian law N_μ on $\mathbb{R}^m$ such that, for any bounded continuous function F on $\mathbb{R}^m$, one has

$$\int_G F\left(\tfrac{\sigma(g,x)-n\lambda}{\sqrt{n}}\right) \mathrm{d}\mu^{*n}(g) \xrightarrow[n\to\infty]{} \int_{\mathbb{R}^m} F(t)\, \mathrm{d}N_\mu(t)\ , \tag{4.22}$$

uniformly for x in X, and

$$\int_G F\left(\tfrac{\kappa(g)-n\lambda}{\sqrt{n}}\right) \mathrm{d}\mu^{*n}(g) \xrightarrow[n\to\infty]{} \int_{\mathbb{R}^m} F(t)\, \mathrm{d}N_\mu(t)\ . \tag{4.23}$$

b) When the local fields $\mathbb{K}_i$ are equal to $\mathbb{R}$ and when μ is supported by $\mathrm{SL}(V_1) \times \ldots \times \mathrm{SL}(V_m)$, the support of this Gaussian law N_μ is the vector subspace E_μ of $\mathbb{R}^m$ spanned by $\ell(G_\mu)$ where G_μ is the Zariski closure of Γ_μ.
c) When $m = 1$, $\mathbb{K}_1 = \mathbb{R}$, and Γ_μ has unbounded image in $\mathrm{PGL}(V_1)$, the gaussian law N_μ is non-degenerate.

Remark 4.12. Point *b*) gives a very practical way to determine the support of the limit gaussian law N_μ. We recall that the *Zariski closure* G_μ of Γ_μ in G is the smallest subset of G containing Γ_μ which is defined by polynomial equations. We recall also that the Zariski closure of a subsemigroup of G is always a group.

Proof. *a*) We first notice that Equations (4.22) and (4.23) are equivalent since, for all $\varepsilon > 0$, there exists $c > 0$ such that, for all non-zero vector v_i in V_i, all $n \geq 1$,

$$\mu^{*n}(\{g \in G \mid c\,\|g_i\| \leq \|g_i v_i\|/\|v_i\| \leq \|g_i\|\}) \geq 1 - \varepsilon.$$

(see for instance [8, Lemma 3.2]).

First, assume that, for $1 \leq i \leq m$, Γ_μ is proximal in V_i. In this case, by Proposition 4.9, in each V_i, the norm cocycle is centerable. Hence our cocycle σ is also centerable. Besides, since by Remark 4.10 μ admits a unique stationary probability measure on X, σ has a unique covariance. Equation (4.22) then directly follows from the central limit theorem 3.4.

In general, by Lemma 4.13 below, for any $1 \leq i \leq m$, there exists a positive integer r_i, a number $C_i \geq 1$ and a finite-dimensional $\mathbb{K}_i$-vector space W_i equipped with a strongly irreducible and proximal representation of Γ_μ such that, for any g in Γ_μ, one has

$$C_i^{-1}\|g_i\|_{V_i}^{r_i} \leq \|g_i\|_{W_i} \leq \|g_i\|_{V_i}^{r_i}.$$

Thus, *a*) follows from the proximal case applied to the representations $W_1, \dots, W_m$.

b) We assume now that all the local fields $\mathbb{K}_i$ are equal to $\mathbb{R}$ and that $\det(g_i) = 1$ for all g in Γ_μ. We want to describe the support of the limit gaussian law N_μ. Again, by Lemma 4.13, we can assume that all V_i's are proximal.

According to [4, §4.6], the set $\kappa(\Gamma_\mu)$ remains at bounded distance from the vector space spanned by $\ell(\Gamma_\mu)$. Hence the support of N_μ is included in E_μ.

Conversely, since σ is centerable, by (3.11), the covariance 2-tensor of N_μ is given by the formula, for all $n \geq 1$,

$$\Phi_\mu = \tfrac{1}{n}\textstyle\int_{G\times X}(\sigma(g,x) - \psi(x) + \psi(gx) - n\lambda)^2\,\mathrm{d}\mu^{*n}(g)\,\mathrm{d}\nu(x) \tag{4.24}$$

where ψ is the continuous function in Equation (3.9) and ν is the uniqe μ-stationary probability measure on X. Let $E_{\Phi_\mu} \subset \mathbb{R}^m$ be the linear span of Φ_μ. For all g in the support of μ^{*n} and all x in the support of ν, the element

$$\sigma(g,x) - \psi(x) + \psi(gx) - n\lambda \ \text{ belongs to } \ E_{\Phi_\mu}. \tag{4.25}$$

In particular, let g be an element of Γ_μ which acts in each V_i as a proximal endomorphism and let

$$x^+ = (x_1^+, \dots, x_m^+)$$

where, for any i, x_i^+ is the attractive fixed point of g in $\mathbb{P}(V_i)$. Since x_i^+ is an eigenline for g_i whose eigenvalue has modulus equal to the spectral radius of g_i, we have

$$\sigma(g, x^+) = \ell(g).$$

Since Γ_μ is strongly irreducible in each V_i, for any $x = (x_1, \dots, x_m)$ in X, there exists h in Γ_μ with $g^n h x \xrightarrow[n\to\infty]{} x^+$. In particular, the support of ν contains x^+, so that, applying (4.25) to the point x^+, we get

$$\ell(g) \in \mathbb{Z}\lambda + E_{\Phi_\mu}. \tag{4.26}$$

Now, since the actions on V_i are strongly irreducible, proximal and volume preserving, the Zariski closure G_μ is semisimple. Hence, by [5], there exists a subset Γ_1 of Γ_μ such that, for any i, the elements of Γ_1 act as proximal endomorphisms in V_i and that the closed subgroup of $\mathbb{R}^m$ spanned by the set $\ell(\Gamma_1)$ in $\mathbb{R}^m$ is equal to the vector space E_μ spanned by $\ell(G_\mu)$. Hence, by (4.26) this space E_μ has to be included in E_{Φ_μ} and we are done.

c) The main difference with point *b*) is that the Zariski closure G_μ may not be semisimple. The same argument as in *b*) tells us that $\ell([G_\mu, G_\mu])$ is included in E_{Φ_μ} and, since the image of Γ_μ in $\mathrm{PGL}(V_1)$ is unbounded, the group $[G_\mu, G_\mu]$ is also unbounded and one must have $E_{\Phi_\mu} = \mathbb{R}$. □

To deduce the general case in Theorem 4.11.*a*) from the one where all the V_i are Γ_μ-proximal, we used the following purely algebraic lemma.

Lemma 4.13. *Let $\mathbb{K}$ be a local field, V be a finite-dimensional normed $\mathbb{K}$-vector space and Γ be a strongly irreducible sub-semigroup of $\mathrm{GL}(V)$. Let $r \geq 1$ be the proximal dimension of Γ in V, that is, the least rank of a non-zero element π of the closure*

$$\overline{\mathbb{K}\Gamma} := \{\pi \in \mathrm{End}(V) \mid \pi = \lim_{n\to\infty} \lambda_n g_n \text{ with } \lambda_n \in \mathbb{K}\ ,\ g_n \in \Gamma\}$$

and let $W \subset \wedge^r V$ be the subspace spanned by the lines $\wedge^r \pi(V)$, where π is a rank r element of $\overline{\mathbb{K}\Gamma}$. Then,
a) W admits a largest proper Γ-invariant subspace U.
b) The action of Γ in the quotient $W' := W/U$ is proximal and strongly irreducible.
c) Moreover, there exists $C \geq 1$ such that, for any g in Γ, one has

$$C^{-1}\|g\|^r \leq \|\wedge^r g\|_{W'} \leq \|g\|^r. \tag{4.27}$$

Remark 4.14. In case $\mathbb{K}$ has characteristic 0, the action of Γ in $\wedge^r V$ is semisimple and $W' = W$.

Proof of Lemma 4.13. a) We will prove that W contains a largest proper Γ-invariant subspace and that this space is equal to

$$U := \cap_\pi \mathrm{Ker}_W(\Lambda^r \pi), \text{ where } \pi \text{ runs among all rank } r \text{ elements of } \overline{\mathbb{K}\Gamma}.$$

This space U is clearly Γ-invariant. We have to check that the only Γ-invariant subspace U_1 of W which is not included in U is $U_1 = W$. Let π be a rank r element of $\overline{\mathbb{K}\Gamma}$ such that U_1 is not included in $\mathrm{Ker}(\wedge^r \pi)$. The endomorphism $\wedge^r \pi$ is proximal and one has

$$\wedge^r \pi(U_1) \subset U_1 \,.$$

As $\wedge^r \pi$ has rank one, one has

$$\mathrm{Im}(\wedge^r \pi) \subset U_1 \,.$$

Let π' be any rank r element of $\overline{\mathbb{K}\Gamma}$. Since Γ is irreducible in V, there exists f in Γ such that $\pi' f \pi \neq 0$. As $\pi' f \pi$ also belongs to $\overline{\mathbb{K}\Gamma}$, we get $\mathrm{rk}(\pi' f \pi) = r$ and, since $\wedge^r(\pi' f)$ preserves U_1, one has

$$\mathrm{Im}(\wedge^r \pi') = \mathrm{Im}(\wedge^r(\pi' f \pi)) \subset U_1 \,.$$

Since this holds for any π', by definition of W, we get $U_1 = W$, which should be proved.

b) The above argument proves also that, for any rank r element π of $\overline{\mathbb{K}\Gamma}$, one has

$$\mathrm{Im}(\Lambda^r \pi) = \Lambda^r \pi(W) \;\text{ and }\; \mathrm{Im}(\Lambda^r \pi) \not\subset U \,. \tag{4.28}$$

In particular, the action of Γ in the quotient space $W' := W/U$ is proximal.

Let us prove now that the action of Γ in W' is strongly irreducible. Let $U_1, \dots, U_r$ be subspaces of W, all of them containing U, such that Γ preserves $U_1 \cup \dots \cup U_r$. Since W' is Γ-irreducible, the spaces $U_1, \dots, U_r$ span W. Let $\Delta \subset \Gamma$ be the sub-semigroup

$$\Delta := \{g \in \Gamma \mid g U_i = U_i \;\text{ for all } 1 \leq i \leq r\}.$$

There exists a finite subset $F \subset \Gamma$ such that

$$\Gamma = \Delta F = F \Delta.$$

In particular, since Γ is strongly irreducible in V, so is Δ. Besides, Δ also has proximal dimension r and, since $\overline{\mathbb{K}\Gamma} = \overline{\mathbb{K}\Delta} F$, W is also spanned by the lines $\mathrm{Im}(\Lambda^r \pi)$ for rank r elements π of $\overline{\mathbb{K}\Delta}$. By applying the first part of the proof to Δ, since the Δ-invariant subspaces U_i span W, one of them is equal to W. Therefore, W' is strongly irreducible.

c) We want to prove the bounds (4.27). First, for g in $\mathrm{GL}(V)$, one has $\| \wedge^r g\| \leq \|g\|^r$. As for g in Γ, we have $(\wedge^r g)W = W$ and $(\wedge^r g)U = U$, we get

$$\| \wedge^r g\|_{W'} \leq \|g\|^r .$$

Assume now there exists a sequence (g_n) in Γ with

$$\|g_n\|^{-r} \| \wedge^r g_n\|_{W'} \to 0$$

and let us reach a contradiction. If $\mathbb{K}$ is $\mathbb{R}$, set $\lambda_n = \|g_n\|^{-1}$. In general, pick λ_n in $\mathbb{K}$ such that $\sup_n |\log(|\lambda_n| \|g_n\|)| < \infty$. After extracting a subsequence, we may assume $\lambda_n g_n \to \pi$, where π is a non-zero element of $\overline{\mathbb{K}\Gamma}$. In particular, π has rank $\geq r$ and we have $\lambda_n^r \wedge^r g_n \to \wedge^r \pi$. Thus, since $\|\lambda_n^r \wedge^r g_n\|_{W'} \to 0$, we get $\| \wedge^r \pi\|_{W'} = 0$, that is,

$$\wedge^r \pi(W) \subset U.$$

We argue now as in *a*). Let π' be a rank r element of $\overline{\mathbb{K}\Gamma}$. Since Γ is irreducible in V, there exists f in Γ such that $\pi' f \pi \neq 0$. Since $\pi' f \pi$ has rank at least r, it has rank exactly r and, since $\wedge^r(\pi' f)$ preserves U, one has

$$\mathrm{Im}(\wedge^r \pi') = \mathrm{Im}(\wedge^r(\pi' f \pi)) \subset U.$$

Since this holds for any π', by definition of W, we get $U = W$. Contradiction. □

Example 4.15. *There exists a finitely supported probability measure μ on $\mathrm{SL}(\mathbb{R}^d)$ such that Γ_μ is unbounded and acts irreducibly on $\mathbb{R}^d$, and such that, if we denote by λ_1 its Lyapunov first exponent, the random variables $\frac{\log \|g_n \cdots g_1\| - n\lambda_1}{\sqrt{n}}$ converge in law to a variable which is not Gaussian.*

Note that, according to Theorem 1.1, the action of Γ_μ on $\mathbb{R}^d$ can not be strongly irreducible. In our example, the limit law is the law of a random variable $\sup(\alpha_1(Z), \ldots, \alpha_m(Z))$ where Z is a D-dimensional Gaussian vector and α_i are linear forms on $\mathbb{R}^D$. One can prove that this is a general phenomenon.

Proof of example 4.15. Set $d = 2$ and $\sigma := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. We just choose $g_i = \sigma^{\varepsilon_i} \left(\begin{smallmatrix} e^{x_i} & 0 \\ 0 & e^{-x_i} \end{smallmatrix}\right)$ where ε_i, x_i are independant random variables, ε_i takes equiprobable values in $\{0, 1\}$ and x_i are symmetric and real-valued with the same law $\nu \neq \delta_0$. One can write $g_n \cdots g_1 = \sigma^{\eta_n} \left(\begin{smallmatrix} e^{S_n} & 0 \\ 0 & e^{-S_n} \end{smallmatrix}\right)$ with $\eta_n = \varepsilon_1 + \cdots + \varepsilon_n$ and

$$S_n = x_1 + (-1)^{\varepsilon_1} x_2 + \cdots + (-1)^{\varepsilon_1 + \cdots + \varepsilon_{n-1}} x_n .$$

By the classical CLT, the sequence $\frac{S_n}{\sqrt{n}}$ converges in law to a non-degenerate Gaussian law. Hence the sequence $\frac{1}{\sqrt{n}} \log \|g_n \cdots g_1\| = \frac{|S_n|}{\sqrt{n}}$ converges in law to a non-Gaussian law. □

4.5. **Central limit theorem for semisimple groups.**

In this section, we prove the central limit theorem for random walks on semisimple Lie groups for a law μ whose second moment is finite and such that Γ_μ is Zariski dense.

This central limit theorem 4.16 will only be an intrinsic reformulation of Theorem 4.11. Its main interest is that it describes more clearly the support of the limit gaussian law.

We first recall the standard notations for semisimple real Lie groups. Let G be a semisimple connected linear real Lie group, $\mathfrak{g}$ its Lie algebra, K a maximal compact subgroup of G, $\mathfrak{k}$ its Lie algebra, $\mathfrak{a}$ a Cartan subspace of $\mathfrak{g}$ orthogonal to $\mathfrak{k}$ for the Killing form, and A the subgroup of G, $A := e^{\mathfrak{a}}$. Let $\mathfrak{a}^+$ be a closed Weyl chamber in $\mathfrak{a}$, $\mathfrak{a}^{++}$ the interior of $\mathfrak{a}^+$, $A^+ = e^{\mathfrak{a}^+}$. Let N be the corresponding maximal nilpotent connected subgroup

$$N := \{n \in G \mid \forall H \in \mathfrak{a}^{++} \ , \ \lim_{t\to\infty} e^{-tH} n e^{tH} = 1\}.$$

Let P be the corresponding minimal parabolic subgroup of G, i.e. P is the normalizer of N. Let $X = G/P$ be the flag variety of G.

Using the Iwasawa decomposition $G = KAN$ one defines the *Iwasawa cocycle* $\sigma : G \times X \to \mathfrak{a}$: for g in G and x in X, $\sigma(g, x)$ is the unique element of $\mathfrak{a}$ such that

$$gk \in K e^{\sigma(g,x)} N \quad , \quad \text{for } x = kP \text{ with } k \text{ in } K.$$

Using the Cartan decomposition $G = KA^+K$, one defines the *Cartan projection* $\kappa : G \to \mathfrak{a}^+$: for g in G, $\kappa(g)$ is the unique element of $\mathfrak{a}^+$ such that

$$g \in K e^{\kappa(g)} K.$$

We also define the *Jordan projection* $\ell : G \to \mathfrak{a}$ by

$$\ell(g) := \lim_{n\to\infty} \tfrac{1}{n} \kappa(g^n).$$

Example Before stating the main theorem, let us describe briefly these notions for $G = \mathrm{SL}(d, \mathbb{R})$. We endow $\mathbb{R}^d$ with the standard Euclidean inner product. In this case, one has:
- $G = \{g \in \mathrm{End}(\mathbb{R}^d) \mid \det(g) = 1\}$, $\mathfrak{g} = \{H \in \mathrm{End}(\mathbb{R}^d) \mid \mathrm{tr}(H) = 0\}$,
- $K = \mathrm{SO}(d, \mathbb{R}) = \{g \in G \mid {}^t g g = e\}$, $\mathfrak{k} = \{H \in \mathfrak{g} \mid {}^t H + H = 0\}$,
- $\mathfrak{a} = \{H = \mathrm{diag}(H_1, \dots, H_d) \in \mathfrak{g}\}$, $\mathfrak{a}^+ = \{H \in \mathfrak{a} \ / \ H_1 \geq \cdots \geq H_d\}$,

- $A = \{a = \mathrm{diag}(a_1, \dots, a_d) \in G \mid a_i > 0\}$, $A^+ = \{a \in A \,|\, a_1 \geq \cdots \geq a_d\}$,
- N is the group of upper triangular matrices with 1's on the diagonal,
- P is the group of all upper triangular matrices in G,
- X is the set of flags $x = (V_i)_{0 \leq i \leq d}$ of $\mathbb{R}^d$, i.e. of increasing sequences of vector subspaces V_i with $\dim V_i = i$.
- The i^{th} coordinate $\sigma_i(g, x)$ of the Iwasawa cocycle $\sigma(g, x)$ is the logarithm of the norm of the transformation induced by g between the Euclidean lines $V_i/V_{i-1} \mapsto gV_i/gV_{i-1}$.
- The coordinates $\kappa_i(g)$ of the Cartan projection $\kappa(g)$ are the logarithms of the eigenvalues of $({}^t gg)^{\frac{1}{2}}$ in decreasing order.
- The coordinates $\ell_i(g)$ of the Jordan projection $\ell(g)$ are the logarithms of the moduli of the eigenvalues of g in decreasing order.

Theorem 4.16. *Let μ be a probability measure on the semisimple connected linear real Lie group G. Assume that Γ_μ is Zariski dense in G, and that the second moment $\int_G \|\kappa(g)\|^2 \, \mathrm{d}\mu(g)$ is finite. Then,*
a) The Iwasawa cocycle is centerable.
b) There exist λ in $\mathfrak{a}^{++}$ and a non-degenerate gaussian law N_μ on $\mathfrak{a}$ such that, for any bounded continuous function F on $\mathfrak{a}$, one has

$$\int_G F\left(\tfrac{\sigma(g,x) - n\lambda}{\sqrt{n}}\right) \mathrm{d}\mu^{*n}(g) \xrightarrow[n\to\infty]{} \int_{\mathfrak{a}} F(t) \, \mathrm{d}N_\mu(t) \, , \tag{4.29}$$

uniformly for x in X, and

$$\int_G F\left(\tfrac{\kappa(g) - n\lambda}{\sqrt{n}}\right) \mathrm{d}\mu^{*n}(g) \xrightarrow[n\to\infty]{} \int_{\mathfrak{a}} F(t) \, \mathrm{d}N_\mu(t) \, . \tag{4.30}$$

We recall that this theorem is due to Goldsheid and Guivarc'h in [23] and to Guivarc'h in [30] when μ has a finite exponential moment.

We recall also that the assumption "Γ_μ is Zariski dense in G" means that, "every polynomial function on G which is identically zero on Γ_μ is identically zero on G".

Proof. a) We use the same method as in [4]. There exists a basis $\chi_1, \dots, \chi_m$ of $\mathfrak{a}^*$ and finitely many irreducible proximal representations $(V_1, \rho_1), \dots, (V_m, \rho_m)$ of G endowed with K-invariant norms such that, for all g in G, and $x = hP$ in X,

$$\chi_i(\kappa(g)) = \log \|\rho_i(g)\| \;\text{ and }\; \chi_i(\sigma(g, x)) = \log \tfrac{\|\rho_i(g)v_i\|}{\|v_i\|},$$

where $\mathbb{R}v_i$ is the hPh^{-1}-invariant line in V_i. It follows then from Theorem 4.9 that, for all $i \leq m$, the cocycle $\chi_i \circ \sigma$ is centerable. Hence the Iwasawa cocycle σ is also centerable.

b) Using the same argument as in a), the convergences to a normal law N_μ in (4.29) and (4.30) follow from Theorem 4.11. This theorem 4.11 tells us also that the support of N_μ is the vector subspace of $\mathfrak{a}$

spanned by the set $\ell(G)$. Since it contains $\mathfrak{a}^+ = \ell(A^+)$, this vector subspace is equal to $\mathfrak{a}$. □

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