On the rate of exponential decay of coefficients on homogeneous spaces

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Abstract

For any homogeneous space of a noncompact semisimple Lie group G, we define an exponent with multiple interpretations from representation theory and group theory. As an application, we give a temperedness criterion for $\mathrm{L}^2(G/H)$ for any closed subgroup H of G, which extends the existing ones of Benoist–Kobayashi for connected subgroups and Lutsko–Weich–Wolf for discrete subgroups.

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1 Introduction

Let G be a connected noncompact semisimple Lie group with finite center, K be a maximal compact subgroup of G, and H be a closed subgroup of G. The homogeneous space G/H admits a G-quasi-invariant Radon measure, giving rise to the unitary representation $\lambda_{G/H}$ of G on the Hilbert space $\mathrm{L}^2(G/H)$ of square-integrable functions on G/H by left translation. These are called the *quasi-regular representations* of G.

1.1 Motivations

The Plancherel formula. The Plancherel formula for a unitary representation of G is the explicit decomposition into irreducible ones. This line of study traces back to the pioneering work of Cartan, Weyl, Gelfand, and Harish-Chandra early in the 20th century. Now thanks to the tremendous input of many other mathematicians, the Plancherel formulae for certain classes of quasi-regular representations $\mathrm{L}^2(G/H)$ are known, including the ones on G itself, the Riemannian/affine symmetric spaces, and the real spherical spaces, cf. [32, 12] and the references therein.

However, for more general homogeneous spaces G/H, the aforementioned approaches hardly generalize, and instead of pursuing the full decomposition, one may ask more accessible questions on the support of the $\mathrm{L}^2(G/H)$, e.g. what are the irreducible representations it may contain.

Tempered homogeneous spaces. One line of such research, intiated by Benoist–Kobayashi, is to study when is the quasi-regular representation $\mathrm{L}^2(G/H)$ a tempered representation, or equivalently when is the support of $\mathrm{L}^2(G/H)$ contained in that of $\mathrm{L}^2(G)$. In a series of work [3, 5, 4, 6], they established a simple geometric criterion for those H with finitely many connected components: $\mathrm{L}^2(G/H)$ is tempered iff the Lie algebras $\mathfrak{g}, \mathfrak{h}$ of G, H satisfy a growth condition.

In a complementary direction, the same question for discrete subgroups have been studied by other authors. The case of G being simple of real rank one is well known from hyperbolic geometry: for a discrete subgroup Γ, $\mathrm{L}^2(G/\Gamma)$ is tempered iff the Laplacian on the locally symmetric space $K\backslash G/\Gamma$ has maximal spectral gap. The latter condition is known equivalent to that the critical exponent of Γ does not exceed half of that of G, by the accumulative works of Elstrodt [14], Patterson [26], Sullivan [31], Corlette [7], Shalom [30], and Leuzinger [23].

When the semisimple Lie group G has higher rank, this result has been extended more recently, first by Edwards–Oh [13] and Lee–Oh [22] for Anosov subgroups and then by Lutsko–Weich–Wolf [24] for all discrete subgroups, to a similar growth criterion on the temperedness of $\mathrm{L}^2(G/\Gamma)$, with the critical exponent replaced by Quint's growth indicator function ψ_Γ introduced in [27].

> *The goal of this paper is to unify these geometric criteria for the temperedness of* $\mathrm{L}^2(G/H)$ *and extend them quantitatively to all homogeneous spaces of* G*, i.e. to arbitrary closed subgroups* H*.*

1.2 Notation and reformulation

We fix some notation and then introduce four exponents related to the homogeneous space G/H, which will be the main protagonists of our paper.

They will allow us to rephrase the aforementioned temperedness criteria.

Unless otherwise stated, the Lie algebra of a Lie group will be denoted by the corresponding Fraktur letter. Let $\theta : \mathfrak{g} \to \mathfrak{g}$ be a Cartan involution which fixes $\mathfrak{k}$ and the associated Cartan decomposition be $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Let $\mathfrak{a}$ be a Cartan subspace in $\mathfrak{p}$ (a maximal real split abelian subalgebra). The adjoint action of $\mathfrak{a}$ on $\mathfrak{g}$ is jointly diagonalizable over $\mathbb{R}$, giving rise to the root system Σ with $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha\in\Sigma} \mathfrak{g}_\alpha$. Fix a positive system Σ^+. The real linear form

$$\rho := \frac{1}{2} \sum_{\alpha\in\Sigma^+} (\dim \mathfrak{g}_\alpha)\alpha \in \mathfrak{a}'$$

is defined in the standard way and measures the complexity of the adjoint action. The closed positive additive and multiplicative Weyl chambers are respectively denoted by $\mathfrak{a}^+ := \{X \in \mathfrak{a} : \alpha(X) \geq 0, \forall \alpha \in \Sigma^+\}$ and $A^+ := \exp \mathfrak{a}^+$. We obtain the Cartan decomposition $G = KA^+K$, for which the A^+-component is unique, and thus the Cartan projection

$$\kappa : G \to \mathfrak{a}^+, \quad g \in K \exp \kappa(g) K,$$

which measures how far group elements are from the identity. We will write $\rho\kappa$ for the function $\rho \circ \kappa : G \to \mathbb{R}^+$ for brevity.

Definition 1.1. The *coefficient decay exponent* $\theta_{G/H}$ is defined as the infimum of $\theta \in [0, 1]$ such that for any $f \in C_c(G/H)$, there exists a constant $C = C(f) > 0$ such that uniformly for all $g \in G$, we have

$$\left|\left\langle \lambda_{G/H}(g)f, f\right\rangle\right| \leq Ce^{2(\theta-1)\rho\kappa(g)}.$$

Motivation for $\theta_{G/H}$. By the Cowling–Haagerup–Howe theorem [10], we have (Corollary 3.10)

$$\mathrm{L}^2(G/H) \text{ is tempered} \iff \theta_{G/H} \leq 1/2. \tag{1.1}$$

Remark. The number $\theta_{G/H}$ does not depend on the choice of the Cartan decomposition of G. For a discrete subgroup Γ, the exponent $\theta_{G/\Gamma}$ is related to the number $\theta_\Gamma(\rho)$ defined in [24, §1] via

$$\theta_\Gamma(\rho) = \max\left\{2\theta_{G/\Gamma} - 1,\, 0\right\}.$$

Definition 1.2. The *optimal integrability exponent* $p_{G/H}$ is defined as the infimum of $p \in [1, \infty]$ such that for any $f \in C_c(G/H)$, we have

$$\left\langle \lambda_{G/H}(\cdot)f, f\right\rangle \in \mathrm{L}^p(G).$$

Motivation for $p_{G/H}$. By [10, Thm 1], we have

$$\mathrm{L}^2(G/H) \text{ is tempered} \iff p_{G/H} \leq 2. \tag{1.2}$$

When $p_{G/H} > 2$, the equality $\theta_{G/H} = 1 - 1/p_{G/H}$ is essentially known by combining [29, Thm 5.3] with [11, Cor 4.3].

The exponent $p_{G/H}$ gives a measurement of the spectral gap of $\mathrm{L}^2(G/H)$. Indeed, as a consequence of the Kunze–Stein phenomenon [9], the theorem [29, Thm 5.3] of Samei–Wiersma implies that, when $p_{G/H} \geq 2$, **all** the matrix coefficients of $\mathrm{L}^2(G/H)$ are $\mathrm{L}^{p_{G/H}+\varepsilon}$-functions for all $\varepsilon > 0$, and moreover this property is satisfied by all those unitary representations that are weakly contained in $\mathrm{L}^2(G/H)$. Hence, the integrability exponent $p_{G/H}$ controls the support of $\mathrm{L}^2(G/H)$.

Remark. The optimal integrability exponent coincides with the number $p_{G/H}$ defined in [3, §4.2] when H is reductive and with the number $q(G; G/H)$ defined in [20, Def 7.12] when H is a unimodular subgroup.

Definition 1.3 (Definition 4.12)**.** The *relative volume growth exponent* $\delta_{G/H}$ of H inside G, which turns out to lie in $[0, 1]$, is defined by

$$\delta_{G/H} := \max\left\{0,\ \limsup_{g\to\infty} \frac{\log \nu_H(H \cap BgB)}{\log \nu_G(BgB)}\right\},$$

where B is any compact subset of G of nonempty interior. The measure ν_G is the Haar measure on G, while $\mathrm{d}\nu_H(h) = (\det \mathrm{Ad}_H\, h)^{1/2}\, \mathrm{d}h$ is the symmetric measure on H (see (2.1)).

Motivation for $\delta_{G/H}$. As a measurement of the exponential volume growth rate of H but relative to the ambient group G, the exponent $\delta_{G/H}$ generalizes the critical exponents of discrete subgroups. As one would expect, it equals the abscissa of convergence for the following analogue of Poincaré series (Proposition 4.14)

$$[0, \infty] \ni t \longmapsto \int_H e^{-2t\rho\kappa(h)}\, \mathrm{d}\nu_H(h).$$

For a discrete subgroup Γ, one recovers $\delta_{G/\Gamma}$ as the 2ρ-directional critical exponent (Proposition 4.17). Then [24, Thm 1.1] is translated to

$$\max\{\theta_{G/\Gamma},\ 1/2\} = \max\{\delta_{G/\Gamma},\ 1/2\}, \tag{1.3}$$

which gives the criterion involving Quint's growth indicator function ψ_Γ:

$$\mathrm{L}^2(G/\Gamma) \text{ is tempered} \iff \psi_\Gamma \leq \rho. \tag{1.4}$$

Definition 1.4 (Definition 4.10)**.** The *local volume decay exponent* $\beta_{G/H} \in [0, 1]$ is defined through the Lie algebras $\mathfrak{g}, \mathfrak{h}$ by

$$\beta_{G/H} := \sup_{\mathfrak{h}} \frac{\rho_{\mathfrak{h}}}{\rho_{\mathfrak{g}}},$$

where the rho-functions $\rho_{\mathfrak{h}}, \rho_{\mathfrak{g}} : \mathfrak{h} \to \mathbb{R}^+$ are respectively defined as the half sum of absolute values of the real parts of complex eigenvalues for the adjoint action of $\mathfrak{h}$ on the spaces $\mathfrak{h}, \mathfrak{g}$, as in [5, §2.3].

Motivation for $\beta_{G/H}$. The rho-functions were used by Benoist–Kobayashi in [3] to capture the exponential volume decay rate from algebraic data (Corollary 4.9). They essentially proved for any reductive subgroup H,

$$\theta_{G/H} = \beta_{G/H} = 1 - \frac{1}{p_{G/H}}. \tag{1.5}$$

In [5], they extended the temperedness criterion to the statement that for any closed subgroup H with finitely many connected components,

$$\mathrm{L}^2(G/H) \text{ is tempered } \iff \beta_{G/H} \leq 1/2. \tag{1.6}$$

Our main results establish precise quantitative relations among the four exponents $\theta_{G/H}, \delta_{G/H}, p_{G/H}, \beta_{G/H}$ that we have just defined.

1.3 Statement of main results

Let G be a noncompact real semisimple algebraic group. Our first main result contains a response to the optimal integrability problem [20, Prob 7.13] for all homogeneous spaces of G.

Theorem A. *Let H be a closed subgroup of G. Then*

$$\theta_{G/H} = \delta_{G/H} = 1 - \frac{1}{p_{G/H}} \geq \beta_{G/H}.$$

Our proof of the equality $\theta_{G/H} = \delta_{G/H}$ is inspired by the method of [24] which we generalize to all induced representations.

As an immediate consequence of Theorem A and the uniform decay characterization (1.1) of temperedness, we obtain the following temperedness criterion, in response to [20, Prob 7.18].

Corollary B. *Let H be a closed subgroup of G. Then*

$$\mathrm{L}^2(G/H) \text{ is tempered } \iff \delta_{G/H} \leq 1/2.$$

In addition, Theorem A unifies the results (1.5) of Benoist–Kobayashi and (1.3) of Lutsko–Weich–Wolf for the following reason: in special cases, one can easily relate the relative volume growth exponent $\delta_{G/H}$ to other existing quantities and obtain the following corollaries.

Corollary C. *Let H be a reductive subgroup of G. Then*

$$\theta_{G/H} = \delta_{G/H} = \beta_{G/H}.$$

Corollary D. *Let Γ be a discrete subgroup of G. Then*

$$\theta_{G/\Gamma} = \delta_{G/\Gamma} = \max\left\{ \sup_{\mathfrak{a}^+} \frac{\psi_\Gamma}{2\rho}, 0 \right\}.$$

Our second main result, concerning closed subgroups with finitely many connected components, gives a quantitative extension of the temperedness criterion (1.6) of Benoist–Kobayashi.

Proposition E. *Let H be a closed subgroup of G with finitely many connected components. If $\theta_{G/H} > 1/2$, then $\beta_{G/H} = \theta_{G/H}$.*

Remark. (1) This statement is sharp in the sense that it fails for values below $1/2$ without further assumptions, as we explain in Example 4.19.

(2) Combining Theorem A and Proposition E, one sees that for closed subgroups of G with finitely many components, the two geometric exponents $\delta_{G/H}$ and $\beta_{G/H}$ are equal if $\delta_{G/H} > 1/2$.

Our proof uses unitary representation theory which only gives information above $1/2$ (i.e. when $\mathrm{L}^2(G/H)$ is non-tempered). We apply the same strategy as Theorem A, along with the extra input of spherical functions and ingredients borrowed from [5].

Finally, we present the following result of independent interest, to be compared with the existing statements on discrete subgroups in [28, 8, 16].

Proposition F (Proposition 6.16)**.** *For any closed subgroups $F < H < G$ with $\mathrm{L}^2(G/H)$ non-tempered (or equiv. $\delta_{G/H} > 1/2$), if F is co-amenable in H, then $\delta_{G/F} = \delta_{G/H}$.*

1.4 Organization

In Section 2, we recall some elements in the theory of semisimple Lie groups. In Section 3, we recall some definitions and facts about unitary representations. In Section 4, we establish the fundamental tools to address the growth and decay of volume in real semisimple groups, which provide the machinery for the main proofs. In Section 5, we prove Theorem A and then deduce Corollary C and Corollary D. In Section 6, we prove Proposition E.

2 Analysis on semisimple Lie groups

2.1 Measures on homogeneous spaces

Let G be a locally compact group and $\mathrm{d}x$ be a left Haar measure on G. The convention for the modular function $\Delta_G : G \to \mathbb{R}_{>0}$ depends on the author. Here we define it as the continuous group morphism such that

$$\int_G f(xg^{-1})\,\mathrm{d}x = \Delta_G(g)\int_G f(x)\,\mathrm{d}x, \quad \text{for all } f \in C_{\mathrm{c}}(G),$$

or formally $\mathrm{d}(xg) = \Delta_G(g)\,\mathrm{d}x$. If G is a Lie group and $\mathrm{Ad}_G : G \to \mathrm{GL}(\mathfrak{g})$ is the adjoint representation, then we have $\Delta_G(g) = \det \mathrm{Ad}_G(g)^{-1}$.

A locally compact group G is unimodular if $\Delta_G \equiv 1$. In general, a right Haar measure on G can be defined by $\mathrm{d}(x^{-1}) = \Delta_G(x)^{-1}\,\mathrm{d}x$. What will play a role later is the *symmetric measure* ν_G on G defined by

$$\mathrm{d}\nu_G(x) := \Delta_G(x)^{-\frac{1}{2}}\,\mathrm{d}x. \tag{2.1}$$

The symmetry comes from the observation that $\mathrm{d}\nu_G(x^{-1}) = \mathrm{d}\nu_G(x)$.

Let H be a closed subgroup of the locally compact group G. The quasi-invariant measures on the homogeneous space G/H are characterized by the following lemma. Be noted that the integration formula holds up to normalization of the Haar measures. On locally compact groups, we take the left Haar measures unless otherwise stated.

Lemma 2.1 ([1, Lem B.1.3])**.** *The homogeneous space G/H always admits a G-quasi-invariant Radon measure. More precisely, the following data are equivalent:*

(1) a density function $\delta : G \to \mathbb{R}_{>0}$ which is continuous and satisfies

$$\delta(gh) = \frac{\Delta_H(h)}{\Delta_G(h)}\delta(g) \tag{2.2}$$

for all $g \in G$ and $h \in H$;

(2) a quasi-invariant Radon measure μ on G/H.

The connection between these two items is given by

$$\int_G f(g)\delta(g)\,\mathrm{d}g = \int_{G/H}\int_H f(gh)\,\mathrm{d}h\,\mathrm{d}\mu(gH) \tag{2.3}$$

for all $f \in C_{\mathrm{c}}(G)$. Moreover, the Radon–Nikodym derivative is given explicitly for all $g \in G$ and $x \in G$ by

$$\frac{\mathrm{d}(g_*\mu)}{\mathrm{d}\mu}(xH) = \frac{\delta(g^{-1}x)}{\delta(x)}. \qquad \square$$

In particular, when there exists a G-invariant Radon measure on G/H, i.e. when $\Delta_G|_H \equiv \Delta_H$, such a measure is unique up to scalar. The following lemma serves as a tool to deduce integration formulae on Lie groups, while being general itself.

Lemma 2.2 ([19, Prop 5.26])**.** *Let S, T be closed subgroups of G so that the complement of ST in G has zero Haar measure, while $K = S \cap T$ is a compact subgroup. Then we can normalize the Haar measures so that for all $f \in C_{\mathrm{c}}(G)$ we have*

$$\int_G f(g)\,\mathrm{d}g = \int_S\int_T f(st)\frac{\Delta_G(t)}{\Delta_T(t)}\,\mathrm{d}s\,\mathrm{d}t. \qquad \square$$

2.2 Semisimple groups and parabolic subgroups

The general references for this part include [19, 18].

For the rest of this section, let G denote a noncompact semisimple real algebraic group, but the results to be discussed are valid for real reductive groups with mild modifications. The rich structure theory of G gives rise to a variety of integration formulae. We continue with the notation introduced in Section 1.2. The semisimple Lie algebra $\mathfrak{g}$ admits an adjoint-invariant inner product $B_\theta(X,Y) = -B_0(X,\theta Y)$, where $B_0(X,Y) := \operatorname{tr}(\operatorname{ad} X \operatorname{ad} Y)$ is the Killing form of $\mathfrak{g}$.

Let $\mathfrak{m} := Z_{\mathfrak{k}}(\mathfrak{a})$ be the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. In the root space decomposition, we have $\mathfrak{g}_0 = \mathfrak{m} \oplus \mathfrak{a}$. Let M and M' be respectively the centralizer and normalizer group of A in K. Then $\operatorname{Lie}(M) = \operatorname{Lie}(M') = \mathfrak{m}$ and the finite quotient group $M'/M = W(\mathfrak{a};\mathfrak{g}) =: W_G$ is the (restricted) Weyl group which acts simply transitively on the set of Weyl chambers of $\mathfrak{a}$.

Let $\mathfrak{n} = \oplus_{\alpha\in\Sigma^+}\mathfrak{g}_\alpha$ and N be the associated analytic subgroup of G. Then we have the Iwasawa decomposition $G = KAN$ where each component is uniquely defined. Denote the Iwasawa projection by

$$\eta : G \to \mathfrak{a}, \quad g \in Ke^{\eta(g)}N.$$

2.2.1 The Cartan projection. The inner product B_θ allows one to identify $\mathfrak{a}$ with the dual space $\mathfrak{a}'$. We write $|\cdot|$ for the corresponding Euclidean norm on $\mathfrak{a}$. This norm is W_G-invariant by the ad-invariance of B_θ. Denote by $\mathfrak{a}(r)$ the closed metric ball centered at 0 of radius r.

Recall that $\kappa : G \to \mathfrak{a}^+$ denotes the Cartan projection. We say that a sequence $(g_n)_{n\in\mathbb{N}}$ of elements in G *go to infinity* (write $g_n \to \infty$), if they eventually leave every compactum of G. This is equivalent to saying that $|\kappa(g_n)| \to +\infty$ as $n \to \infty$. This notion is compatible with group translation on both left and right, as is shown by the following lemma.

Lemma 2.3 ([2, Prop 5.1])**.** *For any compact subset B of G, there exists $r = r(B) > 0$ such that $\kappa(BgB) \subset \kappa(g) + \mathfrak{a}(r)$ for all $g \in G$.* □

2.2.2 Parabolic subgroups. The closed subgroup $Q_0 := MAN$ is a minimal parabolic subgroup of G. Let Q be a parabolic subgroup of G containing Q_0 and $Q = M_Q A_Q N_Q$ be the Langlands decomposition of Q, cf. [19, §V.5]. The subgroup M_Q is reductive and the subgroup $L := M_Q A_Q$ is a Levi factor of the parabolic subgroup Q with $Q = L \ltimes N_Q$.

Let $\Pi' \subset \Pi$ be the subset of simple positive roots orthogonal to the subspace $\mathfrak{a}_Q$. Write $\langle\Pi'\rangle$ for the span of Π'. Dually, $\mathfrak{a}_Q$ is the orthogonal complement of $\langle\Pi'\rangle$. Define $\Sigma_Q^+ := \Sigma^+ \setminus \langle\Pi'\rangle$ and $\Sigma_M^+ := \Sigma^+ \cap \langle\Pi'\rangle$. With the Euclidean structure on $\mathfrak{a}$ induced by the Killing form, let

$$\mathfrak{a}_M := \mathfrak{a}_Q^\perp \text{ in } \mathfrak{a}, \quad \mathfrak{n}_M := \bigoplus_{\alpha\in\Sigma_M^+} \mathfrak{g}_\alpha.$$

Then as vector spaces, we have

$$\mathfrak{m}_Q = \mathfrak{m} \oplus \mathfrak{a}_M \oplus \mathfrak{n}_M \oplus \theta\mathfrak{n}_M, \quad \mathfrak{n}_Q = \bigoplus_{\alpha \in \Sigma_Q^+} \mathfrak{g}_\alpha, \quad \mathfrak{a} = \mathfrak{a}_M \oplus \mathfrak{a}_Q, \quad \mathfrak{n} = \mathfrak{n}_M \oplus \mathfrak{n}_Q.$$

Let $K_M = K \cap M_Q$ and A_M, N_M be the analytic subgroups corresponding to $\mathfrak{a}_M, \mathfrak{n}_M$. Then $M_Q = K_M A_M N_M$ is an Iwasawa decomposition of M_Q, $A = A_M A_Q \cong A_M \times A_Q$, and $N = N_M N_Q \cong N_M \ltimes N_Q$. The group M_Q centralizes A_Q and normalizes N_Q. We remark that all the groups discussed here are closed subgroups of G.

Notation. Define the following real linear forms on $\mathfrak{a}$:

$$\rho_Q = \frac{1}{2} \sum_{\alpha \in \Sigma_Q^+} (\dim \mathfrak{g}_\alpha)\alpha, \quad \rho_M = \frac{1}{2} \sum_{\alpha \in \Sigma_M^+} (\dim \mathfrak{g}_\alpha)\alpha,$$

so that $\rho = \rho_Q + \rho_M$. Sometimes ρ_M is denoted by ρ_L. For $X \in \mathfrak{a}$, write X_Q, X_M respectively for the orthogonal projection of X to the subspaces $\mathfrak{a}_Q, \mathfrak{a}_M$.

Recall that $\eta : G \to \mathfrak{a}$ denotes the Iwasawa projection. The modular function of a parabolic subgroup Q can be computed from the adjoint action: we have $\Delta_Q(q) = e^{-2\rho_Q \eta(q)}$ and in particular $\Delta_{Q_0}(q) = e^{-2\rho\eta(q)}$. Remark that the symmetric measure on Q is thus given from (2.1) by $\mathrm{d}\nu_Q(q) = e^{\rho_Q \eta(q)}\,\mathrm{d}q$.

Lemma 2.4. *For every $X \in \mathfrak{a}$, we have $\rho_Q(X_M) = \rho_M(X_Q) = 0$.*

Proof. That $\rho_M(X_Q) = 0$ follows directly from the definition. To prove $\rho_Q(X_M) = 0$, let us assume that $\Pi' \neq \emptyset$; otherwise, there is nothing to prove. Dually, this is equivalent to $\alpha \perp \rho_Q$ for all $\alpha \in \Pi'$. But any $\alpha \in \Pi'$ is a simple root, so the α-reflection s_α preserves setwise $\Sigma \cap \langle \Pi' \rangle$ and hence also $\Sigma^+ \setminus \langle \Pi' \rangle$. But that means $s_\alpha(\rho_Q) = \rho_Q$, i.e. $\alpha \perp \rho_Q$. □

Lemma 2.5. *The map $\rho_Q \circ \eta : G \to \mathbb{R}$ is K_M-right-invariant.*

Proof. Since the group M_Q normalizes both A_Q and N_Q, Lemma 2.4 implies $\rho_Q\eta(kman) = \rho_Q(\log a)$ for any $k \in K$, $m \in M_Q$, $a \in A_Q$, and $n \in N_Q$. For $k_M \in K_M$, we have $kmank_M = kmk_M an'$ where $n' = k_M^{-1} n k_M \in N_Q$. Hence, we have $\rho_Q\eta(kmank_M) = \rho_Q\eta(kman)$. □

2.2.3 Integral formulae. Lemma 2.2 implies the following formula.

Lemma 2.6 (Integration from Iwasawa decomposition)**.** *Let Q be a parabolic subgroup as before. For any integrable function f on G, we have*

$$\int_G f(g)\,\mathrm{d}g = \int_K \int_Q f(kq)\, e^{2\rho_Q \eta(q)}\,\mathrm{d}k\,\mathrm{d}q. \qquad \square$$

Lemma 2.7 (Integration from Cartan decomposition). *For any integrable function f on G, we have*

$$\int_G f(g)\,\mathrm{d}g = \int_K \int_{\mathfrak{a}^+} \int_K f(k_1 e^X k_2) \left(\prod_{\alpha\in\Sigma^+} \sinh^{\dim \mathfrak{g}_\alpha} \alpha(X) \right) \mathrm{d}k_1\,\mathrm{d}X\,\mathrm{d}k_2. \quad \square$$

For each group element $w \in W_G$, fix a representative $m_w \in M'$ and write $N^w := m_w N m_w^{-1}$ which is independent of the choice of m_w. Let w^* be the unique element of W which maps $\mathfrak{a}^+$ to $-\mathfrak{a}^+$. Denote by $\overline{N} := N^{w^*} = \Theta N$.

Lemma 2.8 (Bruhat decomposition). *We have the following decomposition*

$$G = \bigsqcup_{w\in W_G} Q_0 m_w Q_0 = \bigsqcup_{w\in W_G} MANN^w m_w,$$

where the double class $Q_0 m_{w^*} Q_0 = m_{w^*}\overline{N}MAN$ is an open submanifold of G of full Haar measure, while the other double classes are submanifolds of strictly lower dimensions. $\square$

Hence, $\overline{N}MAN$ is an open submanifold of G whose complement has zero Lebesgue measure. Moreover, multiplication map $\overline{N} \times MAN \to \overline{N}MAN$ is a diffeomorphism. We also have the following formula by Lemma 2.2.

Lemma 2.9 (Integration from Bruhat decomposition). *For any integrable function f on G, we have*

$$\int_G f(g)\,\mathrm{d}g = \int_{\overline{N}} \int_M \int_A \int_N f(\bar{n}man)\, e^{2\rho\log a}\,\mathrm{d}\bar{n}\,\mathrm{d}m\,\mathrm{d}a\,\mathrm{d}n. \quad \square$$

2.3 Spherical functions

To each real linear form $\chi \in \mathfrak{a}'$, we associate the following function on G:

$$\Xi_\chi^G(g) := \int_K e^{-(\chi+\rho)\left(\eta(g^{-1}k)\right)}\,\mathrm{d}k,$$

where $\eta : G \to \mathfrak{a}$ is the Iwasawa projection. These smooth K-bi-invariant functions are *spherical functions* for the pair (G, K). We have the following invariance property of the parameter χ.

Lemma 2.10 ([19, Prop 7.15]). *For any $\chi \in \mathfrak{a}'$ and $w \in W_G$, we have $\Xi_\chi^G = \Xi_{w\chi}^G$.* $\square$

Under the natural identification between the Cartan subspace $\mathfrak{a}$ and $\mathfrak{a}'$, let the positive Weyl chamber $\mathfrak{a}^+$ correspond to $(\mathfrak{a}')^+$. For those χ in the interior of the convex hull of the W-orbit of ρ, the spherical functions decay exponentially fast. In fact, their precise asymptotics can be determined, cf. [25, Thm 3.4]. We will only need the following estimates which are more classical, cf. [19, §VII.8-9] and [17, Exer IV.B.1].

Lemma 2.11. *For each $\chi \in (\mathfrak{a}')^+$, there exists a polynomial $p(\cdot)$ on $\mathfrak{a}$ such that for all $g \in G$, we have*

$$\exp\{(\chi-\rho)\kappa(g)\} \leq \Xi_\chi^G(g) \leq p(\kappa(g))\exp\{(\chi-\rho)\kappa(g)\}\,. \qquad \square$$

3 Unitary representations

The general references include [19, 1].

Let G be a locally compact group. A *unitary representation* $(\pi, \mathcal{H})$ of G consists of $\mathcal{H}$ a complex Hilbert space and $\pi : G \to \mathrm{U}(\mathcal{H})$ a group morphism from G to the group $\mathrm{U}(\mathcal{H})$ of unitary operators on $\mathcal{H}$ which is *strongly continuous* in that for any $v \in \mathcal{H}$, the map $G \to \mathcal{H}$, $g \mapsto \pi(g)v$ is continuous. A *matrix coefficient* of π is a map of the form

$$G \to \mathbb{C}, \quad g \mapsto \langle \pi(g)v_1, v_2\rangle\,,$$

where $v_1, v_2 \in \mathcal{H}$. By strong continuity, matrix coefficients are bounded continuous functions on G.

Two unitary representations $(\pi_1, \mathcal{H}_1)$ and $(\pi_2, \mathcal{H}_2)$ are *unitarily equivalent* if there exists a G-intertwining unitary isomorphism between $\mathcal{H}_1$ and $\mathcal{H}_2$, in which case π_1, π_2 have the same collection of matrix coefficients.

Example. Let $\mathrm{d}x$ be a left Haar measure on G. The convention for the L^2-scalar product of functions is

$$\langle f_1, f_2\rangle = \int_G f_1(x)\overline{f_2(x)}\,\mathrm{d}x.$$

The *left regular representation* λ_G is the unitary representation of G on the Hilbert space $\mathrm{L}^2(G)$ acting by

$$\lambda_G(g)f : x \mapsto f(g^{-1}x), \quad \text{for } f \in \mathrm{L}^2(G).$$

3.1 Induced representations

Let G be a locally compact group, H be a closed subgroup of G, and $(\sigma, \mathcal{V})$ be a unitary representation of H. Let μ be a quasi-invariant Radon measure on the homogeneous space G/H and δ be the associated function satisfying (2.2) (cf. Lemma 2.1).

We describe the induced unitary representation $(\pi, \mathcal{H}) := \mathrm{Ind}_H^G(\sigma, \mathcal{V})$. Elements of $\mathcal{H}$ are measurable vector-valued functions $f : G \to \mathcal{V}$ with σ-equivariance $f(xh) = \sigma(h)^{-1}f(x)$ for all $x \in G$ and $h \in H$, and the L^2-integrability

$$\|f\|^2 := \int_{G/H} \langle f(x), f(x)\rangle_{\mathcal{V}}\,\mathrm{d}\mu(xH) < +\infty,$$

where $\langle f(x), f(x)\rangle_{\mathcal{V}}$ does not depend on the representative of xH since σ is unitary. The induced action of G is given by

$$\pi(g)f(x) = f(g^{-1}x)\left(\frac{\mathrm{d}g_*\mu}{\mathrm{d}\mu}(xH)\right)^{\frac{1}{2}} = f(g^{-1}x)\left(\frac{\delta(g^{-1}x)}{\delta(x)}\right)^{\frac{1}{2}}.$$

Here, the cocycle term ensures that $\pi(g)$ is a unitary operator. Although *a priori* this definition depends on the measure μ, it turns out that different choices of μ give unitarily equivalent representations. In particular, if $\sigma = 1_H$, then π is the *quasi-regular representation* $\lambda_{G/H}$ on $\mathrm{L}^2(G/H)$.

There is a simple way to produce elements in $\mathcal{H}$. For $\varphi \in C_{\mathrm{c}}(G)$ and $v \in \mathcal{V}$, define the map $\mathrm{I}_H^G(\varphi, v) = \mathrm{I}(\varphi, v) : G \to \mathcal{V}$ by

$$\mathrm{I}(\varphi, v)(x) := \int_H \varphi(xh)\sigma(h)v \,\mathrm{d}h. \tag{3.1}$$

We only specify I_H^G when necessary. The equivariance property follows from

$$\mathrm{I}(\varphi, v)(xh_0) = \int_H \varphi(xh_0h)\sigma(h)v \,\mathrm{d}h = \sigma(h_0)^{-1}\mathrm{I}(\varphi, v)(x)$$

for all $x \in G$ and $h_0 \in H$, and the L^2-integrability from

$$\begin{aligned}\|\mathrm{I}(\varphi, v)\|^2 &= \int_{G/H}\left\|\int_H \varphi(xh)\sigma(h)v \,\mathrm{d}h\right\|_{\mathcal{V}}^2 \mathrm{d}\mu(xH)\\ &\le \|v\|_{\mathcal{V}}^2 \int_{G/H}\int_H |\varphi(xh)|^2 \,\mathrm{d}h \,\mathrm{d}\mu(xH)\\ &= \|v\|_{\mathcal{V}}^2 \int_G |\varphi(g)|^2\delta(g) \,\mathrm{d}g < +\infty.\end{aligned}$$

Hence, the function $\mathrm{I}(\varphi, v)$ belongs to $\mathcal{H}$. We have the following fact.

Lemma 3.1 ([1, Lem B.1.2])**.** *Let $\sigma = 1_H$. Then the map $C_{\mathrm{c}}(G) \to C_{\mathrm{c}}(G/H)$ given by $\varphi \mapsto \mathrm{I}(\varphi, 1)$ is surjective.* □

Lemma 3.2. *Given any neighborhood B_G of e in G and any $\psi \in C_{\mathrm{c}}(G)$, there exists finitely many $\varphi_i \in C_{\mathrm{c}}(G)$ with $(\operatorname{supp}\varphi_i)(\operatorname{supp}\varphi_i)^{-1} \subset B_G$ such that $\psi = \sum_i \varphi_i$.*

Proof. Since B_G is a neighborhood of e, we can find a relatively compact open neighborhood B of e with $BB^{-1} \subset B_G$. Now $\{Bg : g \in \operatorname{supp}\psi\}$ gives an open cover of $\operatorname{supp}\psi$. By the compactness of $\operatorname{supp}\psi$, there exists a finite subcover $\{Bg_i : i \in I\}$ for some finite index set I. Then we can find a finite partition of unity $\{\chi_i \in C_{\mathrm{c}}(Bg_i)\}_{i\in I}$ subordinated to this cover, so that $1 = \sum_{i\in I}\chi_i$ over $\operatorname{supp}\psi$. Then $\varphi_i := \psi\chi_i \in C_{\mathrm{c}}(G)$ satisfies $(\operatorname{supp}\varphi_i)(\operatorname{supp}\varphi_i)^{-1} \subset B_G$ and $\psi = \sum_i \varphi_i$. □

Corollary 3.3. *Let $\sigma = 1_H$. For any neighborhood B_G of e in G, the set*

$$\left\{\mathrm{I}(\varphi, 1) : \varphi \in C_{\mathrm{c}}(G) \text{ with } (\operatorname{supp}\varphi)(\operatorname{supp}\varphi)^{-1} \subset B_G\right\}$$

spans $C_{\mathrm{c}}(G/H)$ and hence is a total subset in $\mathrm{L}^2(G/H)$.

Proof. The corollary follows from Lemma 3.1 and Lemma 3.2. □

Lemma 3.4 (Induction by stage [1, Thm E.2.4])**.** *Let $F < H < G$ be two closed subgroups of G. Then for any unitary representation σ of F, we have $\operatorname{Ind}_F^G \sigma \cong \operatorname{Ind}_H^G(\operatorname{Ind}_F^H \sigma)$.* □

3.2 Weak containment

We introduce the notion of weak containment to study unitary representations of noncompact groups. Let G be a lcsc group.

Definition 3.5. Let $(\sigma, \mathcal{V})$ and $(\pi, \mathcal{H})$ be two unitary representations of G. Say that σ is *weakly contained* in π (write $\sigma \prec \pi$), if every diagonal matrix coefficient $\langle \sigma(\cdot)v, v\rangle$ can be approximated, uniformly on compacta, by convex combinations of diagonal matrix coefficients of π.

Fact 3.6. (1) $\sigma \prec \pi$ iff $\|\sigma(f)\|_{\mathrm{op}} \leq \|\pi(f)\|_{\mathrm{op}}$ for all $f \in C_{\mathrm{c}}(G)$.

(2) Weak containment is preserved under induction and restriction of unitary representations.

(3) A locally compact group is amenable iff the trivial representation is weakly contained in its regular representation.

Example 3.7. Let G be a real semisimple algebraic group and Q_0 be a minimal parabolic group. As Q_0 is amenable, the trivial representation 1_{Q_0} is weakly contained in the regular representation λ_{Q_0}. Hence, $\lambda_{G/Q_0} = \operatorname{Ind}_{Q_0}^G 1_{Q_0}$ is weakly contained in $\lambda_G = \operatorname{Ind}_{Q_0}^G \lambda_{Q_0}$.

3.3 Tempered representations

Now let G be a noncompact real semisimple algebraic group.

Definition 3.8. A unitary representation π of G is *tempered* if π is weakly contained in the regular representation $(\lambda_G, \mathrm{L}^2(G))$.

Example 3.7 implies that $(\lambda_{G/Q_0}, \mathrm{L}^2(G/Q_0))$ is tempered. Note that the matrix coefficient $\langle \lambda_{G/Q_0}(\cdot)1, 1\rangle$ equals the Harish-Chandra spherical function Ξ_0 (Section 2.3). In general, the matrix coefficients of tempered representations are characterized as follows.

Theorem 3.9 ([10])**.** *Let G be a real semisimple algebraic group and K be a maximal compact subgroup of G. Then for any unitary representation $(\pi, \mathcal{H})$ of G, the following statements are equivalent:*

(i) π *is tempered;*

(ii) for all K*-finite vectors* v_1, v_2 *in* $\mathcal{H}$*, we have*

$$|\langle \pi(g)v_1, v_2\rangle| \leq \sqrt{\dim\langle Kv_1\rangle \dim\langle Kv_2\rangle}\, \|v_1\|\, \|v_2\|\, \Xi_0^G(g);$$

(iii) there exists a dense subspace $\mathcal{H}_0$ *of* $\mathcal{H}$*, such that for all* $v \in \mathcal{H}_0$*, the coefficients* $\langle \pi(\cdot)v, v\rangle \in \mathrm{L}^{2+\varepsilon}(G)$ *for any* $\varepsilon > 0$*.* □

In view of Lemma 2.11, the optimal decay of spherical functions is given by Ξ_0^G. Meanwhile, it follows from Lemma 2.11 and Lemma 2.7 that we have $\Xi_0^G \in \mathrm{L}^{2+\varepsilon}(G)$ for any $\varepsilon > 0$. For a closed subgroup H of G, we have the following consequence.

Corollary 3.10. $\mathrm{L}^2(G/H)$ *is tempered iff* $\theta_{G/H} \leq 1/2$*.* □

4 Volume growth and volume decay

The goal of this section is to establish the preliminary tools to study the growth and decay of volume in semisimple Lie groups.

Let G be a real semisimple algebraic group and ν_G be its Haar measure. We will be interested for example in the behavior of the volume of BgB, where B is a fixed compactum while g varies in G. The consideration of such volume functions is classical and appears in the related work [15].

4.1 Local volume decay in G

As a consequence of the Bruhat decomposition (Lemma 2.8), the multiplication map $\overline{N} \times M \times A \times N \to G$ is a diffeomorphism onto an open subset of full measure.

Lemma 4.1. *For any compact subset* B *of* $\overline{N}MAN$*, there exists a constant* $C > 0$ *such that for all* $a \in A^+$ *we have* $\nu_G(aBa^{-1} \cap B) \leq Ce^{-2\rho \log a}$*.*

Proof. By the Bruhat decomposition, there exist compact sets

$$B_{\overline{N}} \subset \overline{N},\ B_A \subset A,\ B_N \subset N$$

such that $B \subset B_{\overline{N}} M B_A B_N$. From Lemma 2.9 we deduce

$$\nu_G\big(aBa^{-1} \cap B\big) \leq \int_{B_{\overline{N}}} \int_M \int_{B_A} \int_{B_N} \mathbb{1}_B(a^{-1}\bar{n}ma_1na)\, e^{2\rho \log a_1}\, \mathrm{d}\bar{n}\, \mathrm{d}m\, \mathrm{d}a_1\, \mathrm{d}n.$$

Since a normalizes both $\overline{N}$ and N, and since MA centralizes a, we have furthermore

$$\begin{aligned}\nu_G\big(aBa^{-1}\cap B\big) &\le \int_{B_{\overline{N}}\cap aB_{\overline{N}}a^{-1}} \mathrm{d}\bar{n}\int_M \mathrm{d}m\int_{B_A} e^{2\rho\log a_1}\,\mathrm{d}a_1\int_{B_N}\mathrm{d}n\\ &\le \int_{a^{-1}B_{\overline{N}}a\cap B_{\overline{N}}} e^{-2\rho\log a}\,\mathrm{d}\bar{n}'\,C(B_A,B_N)\\ &\le C(B_{\overline{N}},B_A,B_N)\,e^{-2\rho\log a},\end{aligned}$$

where we set $\bar{n}' = a^{-1}\bar{n}a$ and get $\mathrm{d}\bar{n}' = e^{2\rho\log a}\,\mathrm{d}\bar{n}$. □

Lemma 4.2. *There exists an open neighborhood B_G of e in G, such that for all $k\in K$ we have $kB_Gk^{-1}\subset \overline{N}MAN$.*

Proof. This follows from the openness of $\overline{N}MAN$ in G (Lemma 2.8) and the compactness of K. □

Proposition 4.3. *Let B_G be given as in Lemma 4.2. Then for any functions $\varphi_1,\varphi_2\in C_{\mathrm{c}}(B_G)$, there exists a constant $C = C(\varphi_1,\varphi_2)$ such that uniformly for all $g\in G$,*

$$\left|\int_G \varphi_1(g^{-1}xg)\varphi_2(x)\,\mathrm{d}x\right| \le Ce^{-2\rho\kappa(g)}.$$

Proof. For $i=1,2$, define $\tilde{\varphi}_i(x) := \sup_{k\in K}|\varphi_i(kxk^{-1})|$. Then $\tilde{\varphi}_1,\tilde{\varphi}_2$ are continuous functions compactly supported in $\overline{N}MAN$ by Lemma 4.2. For $g\in G$, we can write $g = k_2e^Xk_1$ for $X=\kappa(g)$ and some $k_1,k_2\in K$. By the unimodularity of G, we have

$$\begin{aligned}\left|\int_G \varphi_1(g^{-1}xg)\varphi_2(x)\,\mathrm{d}x\right| &= \left|\int_G \varphi_1(k_1^{-1}e^{-X}ye^Xk_1)\varphi_2(k_2yk_2^{-1})\,\mathrm{d}y\right|\\ &\le \int_G \tilde{\varphi}_1(e^{-X}ye^X)\tilde{\varphi}_2(y)\,\mathrm{d}y,\end{aligned}$$

which is bounded from above by $Ce^{-2\rho(X)}$ by Lemma 4.1. □

4.2 The rho-function and volume decay

Let H be a Lie group and $R: H\to \mathrm{GL}(V)$ be a continuous linear representation on a d-dimensional real vector space V. By notation abusing, the differential map $R:\mathfrak{h}\to\mathrm{End}(V)$ is a representation of the Lie algebra $\mathfrak{h}$. To these data we associate the following rho-function $\rho_V:\mathfrak{h}\to\mathbb{R}^+$.

Definition 4.4. For each $Y\in\mathfrak{h}$, the action of $R(Y)$ on $V\otimes\mathbb{C}$ admits a Jordan normal form over $\mathbb{C}$ with diagonal elements $\lambda_1,\dots,\lambda_d$. We define

$$\rho_V(Y) := \frac{1}{2}\sum_{i=1}^{d}\left|\mathrm{Re}\,\lambda_i\right|.$$

It follows from the definition that ρ_V is a continuous homogeneous function which is invariant by the adjoint action of H.

Remark 4.5. Let H be a real algebraic group and R be an algebraic representation over $\mathbb{R}$. Let $\mathfrak{a}_\mathfrak{h}$ be a maximal split abelian subalgebra of $\mathfrak{h}$. Since $R(\mathfrak{a}_\mathfrak{h})$ is a split abelian subalgebra of $\mathrm{End}(V)$, the action of $\mathfrak{a}_\mathfrak{h}$ is jointly diagonalizable over $\mathbb{R}$. Then the restriction $\rho_V|_{\mathfrak{a}_\mathfrak{h}}$ is the half sum of the absolute values of the eigenvalues and therefore is a piecewise linear, continuous, convex, homogeneous function. As V is finite-dimensional, the function ρ_V is uniformly Lipschitz on $\mathfrak{a}_\mathfrak{h}$. If R is faithful, then $\rho_V|_{\mathfrak{a}_\mathfrak{h}}$ is a polyhedral norm on $\mathfrak{a}_\mathfrak{h}$.

By the Jordan decomposition, every element $Y \in \mathfrak{h}$ splits uniquely as a sum of commuting elements $Y = Y_e + Y_h + Y_n$ in $\mathfrak{h}$ with Y_e elliptic, Y_h hyperbolic and Y_n nilpotent. Moreover, Y_h is H-conjugated into $\mathfrak{a}_\mathfrak{h}$. Since $\rho_V(Y) = \rho_V(Y_h)$, the function ρ_V is determined by $\rho_V|_{\mathfrak{a}_\mathfrak{h}}$.

Example 4.6. Let H be a real reductive group and $(R, V) = (\mathrm{Ad}, \mathfrak{h})$ be the adjoint representation. Fix a positive system $\Sigma^+(\mathfrak{a}_\mathfrak{h}; \mathfrak{h})$ and let ρ_H be the usual *half* sum of positive roots. Then the convex function $\rho_\mathfrak{h}$ coincides with the twice of the linear form ρ_H on the positive Weyl chamber $\mathfrak{a}_\mathfrak{h}^+$. If $W_H = W(\mathfrak{a}_\mathfrak{h}; \mathfrak{h})$ denotes the Weyl group, then for all $X \in \mathfrak{a}_\mathfrak{h}$,

$$\rho_\mathfrak{h}(X) = \max_{w \in W_H} 2\rho_H(wX).$$

In particular, $\rho_\mathfrak{h}$ is W_H-invariant.

Let Vol be the Lebesgue measure on the vector space $V \cong \mathbb{R}^d$. The function ρ_V reflects the volume decay of the representation R.

Lemma 4.7 ([5, Lem 2.8])**.** *Let $\mathfrak{a} \subset \mathrm{End}(V)$ be a real split abelian subalgebra. Then for any compact neighborhood B of 0 in V, there exist constants $c, C > 0$ such that uniformly for all $X \in \mathfrak{a}$,*

$$ce^{-\rho_V(X)} \leq e^{-\operatorname{tr} R(X)/2} \operatorname{Vol}(R(\exp X)B \cap B) \leq Ce^{-\rho_V(X)}. \qquad \square$$

Corollary 4.8. *For any compact neighborhood B of 0 in V and any semisimple operator $X \in \mathrm{End}(V)$, there exist constants $c_X, C_X > 0$ such that uniformly for all $t \in \mathbb{R}^+$,*

$$c_X e^{-t\rho_V(X)} \leq e^{-\operatorname{tr}(tX)/2} \operatorname{Vol}(\exp(tX)B \cap B) \leq C_X e^{-t\rho_V(X)}.$$

Proof. One applies Lemma 4.7 and absorbs the elliptic part which corresponds to a rotation on V. $\square$

Lemma 4.7 can be rephrased in terms of unitary representations.

Corollary 4.9. *Let H be a real reductive group and $R : H \to \mathrm{GL}(V)$ be an algebraic linear representation. Consider the unitary representation $(\tau, \mathrm{L}^2(V))$ of the group H derived from R, given by*

$$\tau(h)f(v) = f(R(h)^{-1}v)\,(\det R(h))^{-\frac{1}{2}}\,, \tag{4.1}$$

for $f \in \mathrm{L}^2(V)$ and $v \in V$. Then for any compact neighborhood B of 0 in V, there exist constants $c, C > 0$ such that uniformly for all $h \in H$,

$$ce^{-\rho_V \kappa_H(h)} \le \langle \tau(h)\mathbb{1}_B, \mathbb{1}_B \rangle \le Ce^{-\rho_V \kappa_H(h)},$$

where $\kappa_H : H \to \mathfrak{a}_\mathfrak{h}$ is a Cartan projection.

Proof. Let $H = K_H A_H^+ K_H$ be the Cartan decomposition of H associated with κ_H. The subset $D := R(K_H)B$ is a K_H-left-invariant compact neighborhood of 0 in V. Thus for any $k_1, k_2 \in K_H$ and $X \in \mathfrak{a}_\mathfrak{h}$, we have

$$\left\langle \tau(k_1 e^X k_2)\mathbb{1}_D, \mathbb{1}_D \right\rangle = e^{-\operatorname{tr} R(X)/2} \operatorname{Vol}\big(R(e^X)D \cap D\big)\,,$$

whence we can conclude by applying Lemma 4.7. □

Now we can define the local volume decay exponent of G/H.

Definition 4.10. Let G be a semisimple Lie group and H be a closed subgroup. The *local volume decay exponent* $\beta_{G/H}$ is defined by

$$\beta_{G/H} := \sup_{X \in \mathfrak{h}} \frac{\rho_\mathfrak{h}(X)}{\rho_\mathfrak{g}(X)},$$

where the spaces $\mathfrak{g}, \mathfrak{h}$ are viewed as $\mathfrak{h}$-module through the adjoint action. We take $0/0 = 0$ by convention.

Remark. (1) By definition, the number $\beta_{G/H}$ lies in $[0, 1]$.
(2) If $H_1 < H_2$ are closed subgroups of G, then $\beta_{G/H_1} \le \beta_{G/H_2}$.
(3) If both G and H are algebraic groups, then Remark 4.5 implies

$$\beta_{G/H} = \sup_{X \in \mathfrak{a}_\mathfrak{h}} \frac{\rho_\mathfrak{h}(X)}{\rho_\mathfrak{g}(X)}.$$

4.3 Volume growth in G

By $B \Subset G$ we denote that B be a compact subset of G of nonempty interior.

Proposition 4.11. *For any $B \Subset G$, there exist constants $c, C > 0$ such that uniformly for all $g \in G$,*

$$ce^{2\rho\kappa(g)} \le \nu_G(BgB) \le Ce^{2\rho\kappa(g)}.$$ □

Proof. Let $X = \kappa(g)$. For the upper bound, by Lemma 2.3, there exists $r > 0$ such that $\kappa(BgB) \subset \kappa(g) + \mathfrak{a}(r)$ for all $g \in G$, whence we have $BgB \subset Ke^{X+\mathfrak{a}(r)}K$. By Lemma 2.7, we have uniformly for $g \in G$,

$$\nu_G(BgB) \leq \int_K \int_{X+\mathfrak{a}(r)} \int_K e^{2\rho(Y)} \,\mathrm{d}k \,\mathrm{d}Y \,\mathrm{d}k' \leq C \exp\{2\rho(X)\}\,.$$

For the lower bound, first note that by Lemma 2.3, up to translation we can suppose that B contains a neighborhood of $e \in G$. Then we can find a small neighborhood B' of e with $kB'k^{-1} \subset B$ for all $k \in K$ (cf. Lemma 4.2). Write $g = k_1 e^X k_2$. By the unimodularity of G,

$$\nu_G(BgB) = \nu_G\big((k_1^{-1}Bk_1)e^X(k_2Bk_2^{-1})\big) \geq \nu_G\big(B'e^XB'\big).$$

Up to further shrinking B', we can assume $B' \subset \overline{N}MAN$. By the Bruhat decomposition (Lemma 2.8), there exist compact neighborhoods of e in the respective subgroups $B_{\overline{N}} \Subset \overline{N}$, $B_M \Subset M$, $B_A \Subset A$, and $B_N \Subset N$ such that $B_{\overline{N}}B_MB_AB_N \subset B'$. Hence by the unimodularity of G, we deduce

$$\begin{aligned}
\nu_G\big(B'e^XB'\big) &= \nu_G\big(B'e^XB'e^{-X}\big) \\
&\geq \nu_G\big(B_{\overline{N}}\left(e^XB_{\overline{N}}B_MB_AB_Ne^{-X}\right)\big) \\
&= \nu_G\big(\left(B_{\overline{N}}e^XB_{\overline{N}}e^{-X}\right)B_MB_A\left(e^XB_Ne^{-X}\right)\big).
\end{aligned}$$

By further applying Lemma 2.9, we obtain uniformly for $g \in G$,

$$\begin{aligned}
\nu_G\big(B'e^XB'\big) &\geq c\,\nu_{\overline{N}}\big(B_{\overline{N}}e^XB_{\overline{N}}e^{-X}\big)\,\nu_N\left(e^XB_Ne^{-X}\right) \\
&\geq c\,\nu_{\overline{N}}(B_{\overline{N}})\,e^{2\rho(X)}\nu_N(B_N) = c'e^{2\rho(X)}.
\end{aligned}$$

□

4.4 Relative volume growth of closed subgroups

In this subsection, we introduce the relative volume growth exponent of closed subgroups of the noncompact semisimple Lie group G. As it turns out, for a discrete subgroup, it can be recovered from the growth indicator function, while for a reductive subgroup, we relate this quantity to the Lie algebras, or more precisely to the local volume decay exponent $\beta_{G/H}$.

Recall that the symmetric measure on a locally compact group H is given by $\mathrm{d}\nu_H(h) = \Delta_H(h)^{-\frac{1}{2}}\,\mathrm{d}h$. If the group H is either reductive or discrete, then ν_H coincides with the Haar measure.

Definition 4.12. Define the *relative volume growth exponent* of a closed subgroup H in G by

$$\delta_{G/H} := \max\left\{0,\ \sup_{B \Subset G} \limsup_{g\to\infty} \frac{\log \nu_H(H \cap BgB)}{\log \nu_G(BgB)}\right\},$$

where B ranges over compacta of G of nonempty interior.

Remark 4.13. (1) A finite-covering argument shows that the limsup in the definition does not depend on the choice of B.

(2) One can also restrict $g \to \infty$ to be inside H.

Consider now the following analogue of Poincaré series

$$I(t) := \int_H e^{-2t\rho\kappa(h)}\, \mathrm{d}\nu_H(h). \tag{4.2}$$

The following proposition characterizes the growth exponent $\delta_{G/H}$ as the abscissa of convergence of I.

Proposition 4.14. *For any closed subgroup H of a connected noncompact semisimple Lie group G of finite center, we have*

$$\delta_{G/H} = \inf\left\{t \in [0,\infty] : I(t) < \infty\right\}.$$

Proof. Denote the right hand side by δ_0. First show $\delta_{G/H} \le \delta_0$. Let $B \Subset G$. Then by Lemma 2.3, there exists $R > 0$ such that $\kappa(BgB) \subset \kappa(g) + \mathfrak{a}(R)$ for all $g \in G$, whence for all $h \in H \cap BgB$ we have

$$2\rho\kappa(h) \le 2\rho\kappa(g) + C_1,$$

where the constant $C_1 = \sup_{\mathfrak{a}(R)} 2\rho$ is uniform. Since $I(t) < \infty$ for any $t > \delta_0$, we then have

$$\begin{aligned}\nu_H(H \cap BgB) &\le C_2\, e^{2t\rho\kappa(g)} \int_{H\cap BgB} e^{-2t\rho\kappa(h)}\, \mathrm{d}\nu_H(h)\\ &\le C_2\, e^{2t\rho\kappa(g)} I(t) \le C_3\, e^{2t\rho\kappa(g)},\end{aligned}$$

uniformly for all $g \in G$. From the definition of the growth exponent $\delta_{G/H}$ we deduce that $t \ge \delta_{G/H}$, whence we have $\delta_0 \ge \delta_{G/H}$.

Next we show $\delta_{G/H} \ge \delta_0$. Let $\mathcal{L}$ be a lattice of $\mathfrak{a}$ and $\mathcal{L}^+ = \mathcal{L} \cap \mathfrak{a}^+$. Then there exists some $r > 0$ such that $\mathfrak{a}^+$ is covered by the balls of radius r centered at elements in $\mathcal{L}^+$. Fix any number $t > \delta_{G/H}$ and then fix a small number $\varepsilon > 0$ so that $t - \varepsilon > \delta_{G/H}$.

Define the subset $B = Ke^{\mathfrak{a}(r)}K \Subset G$. Lemma 2.3 yields a constant $c > 0$ such that whenever $h \in Be^X B$, we have $2\rho(X) \le 2\rho\kappa(h) + c$, whence

$$\int_{H\cap Be^X B} e^{-2t\rho\kappa(h)}\, \mathrm{d}\nu_H(h) \le C_4\, e^{-2t\rho(X)} \nu_H\big(H \cap Be^X B\big)$$

uniformly for all $X \in \mathfrak{a}^+$. But since $t - \varepsilon > \delta_{G/H}$, we deduce from Proposition 4.11 that

$$\nu_H\big(H \cap Be^X B\big) \le C_5\, e^{2(t-\varepsilon)\rho(X)}$$

uniformly for $X \in \mathfrak{a}^+$, whence for all $X \in \mathfrak{a}^+$ we have

$$\int_{H\cap Be^X B} e^{-2t\rho\kappa(h)}\, \mathrm{d}\nu_H(h) \le C_6\, e^{-2\varepsilon\rho(X)}.$$

Since the construction implies that the subset $Be^X B$ contains $Ke^{X+\mathfrak{a}(r)}K$ for all $X \in \mathfrak{a}^+$, we obtain

$$I(t) \leq \sum_{X\in\mathcal{L}^+} \int_{H\cap Be^X B} e^{-2t\rho\kappa(h)}\,\mathrm{d}\nu_H(h) \leq C_6 \sum_{X\in\mathcal{L}^+} e^{-2\varepsilon\rho(X)},$$

which is finite as the lattice $\mathcal{L}$ grows polynomially. Hence, we have $t > \delta_0$ and we obtain $\delta_{G/H} \geq \delta_0$. □

4.4.1 Reductive subgroups.

Proposition 4.15. *If H is a reductive subgroup of G, then*

$$\delta_{G/H} = \beta_{G/H}.$$

First we introduce the notation. Let A_H be a Cartan subspace of H. Extending A_H to a Cartan subspace A of G, we have $A_H = A \cap H$. By the reductiveness of H, there exists a Cartan decomposition $G = KAK$ of G such that the subgroup $K_H := K \cap H$ is a maximal compact subgroup of H, with the Cartan decomposition of H given by $H = K_H A_H K_H$.

Let $\mathfrak{a}, \mathfrak{a}_\mathfrak{h}$ denote respectively the Cartan subspaces and W_G, W_H denote respectively the associated Weyl groups. Since the Killing form is adjoint-invariant, the induced Euclidean norm $|\cdot|$ on $\mathfrak{a}$ is W_G-invariant and its restriction to $\mathfrak{a}_\mathfrak{h}$ is W_H-invariant.

By the Cartan decomposition of H, we have for all $h \in H$ that

$$h \in K_H \exp\{\{w\kappa(h)\colon w \in W_G\} \cap \mathfrak{a}_\mathfrak{h}\}\, K_H. \tag{4.3}$$

Lemma 4.16. *For any $B \Subset G$, there exists a constant $C > 0$ such that uniformly for all $g \in G$,*

$$\nu_H(H \cap BgB) \leq Ce^{2\beta_{G/H}\rho\kappa(g)}.$$

Proof. Write $X = \kappa(g)$. By Lemma 2.3, there exists a constant $r > 0$ such that $\kappa(BgB) \subset \kappa(g) + \mathfrak{a}(r)$ for all $g \in G$. Then (4.3) yields

$$H \cap BgB \subset \bigcup_{w\in W_G} K_H e^{(wX+\mathfrak{a}(r))\cap\mathfrak{a}_\mathfrak{h}} K_H. \tag{4.4}$$

Hence, by Lemma 2.7 and Example 4.6, we have

$$\nu_H(H \cap BgB) \leq \sum_{w\in W_G} \int_{(wX+\mathfrak{a}(r))\cap\mathfrak{a}_\mathfrak{h}} \exp\{\rho_\mathfrak{h}(Y)\}\,\mathrm{d}Y. \tag{4.5}$$

Since $\rho_\mathfrak{h} \leq \beta_{G/H}\rho_\mathfrak{g}$ by the definition of $\beta_{G/H}$, and since $\rho_\mathfrak{g}$ is W_G-invariant and uniformly Lipschitz on $\mathfrak{a}_\mathfrak{h}$, we have

$$\begin{aligned}\exp\{\rho_\mathfrak{h}(Y)\} &\leq \exp\big\{\beta_{G/H}\rho_\mathfrak{g}(Y)\big\} \leq C_1 \exp\big\{\beta_{G/H}\rho_\mathfrak{g}(wX)\big\}\\ &= C_1 \exp\big\{\beta_{G/H}\rho_\mathfrak{g}(X)\big\} = C_1 \exp\big\{2\beta_{G/H}\rho(X)\big\}\end{aligned}$$

uniformly for $Y \in wX + \mathfrak{a}(r)$. We conclude by feeding back to (4.5). □

Proof of Proposition 4.15. For $B \Subset G$, Proposition 4.11 and Lemma 4.16 yield

$$\frac{\log \nu_H(H \cap BgB)}{\log \nu_G(BgB)} \leq \frac{C_1 + 2\beta_{G/H}\rho\kappa(g)}{C_2 + 2\rho\kappa(g)}$$

uniformly for all $g \in G$. Since $\rho\kappa(g) \to \infty$ as $g \to \infty$, we get $\delta_{G/H} \leq \beta_{G/H}$.

To show $\delta_{G/H} \geq \beta_{G/H}$, let $B \Subset G$ and $B_H := B \cap H \Subset H$. Then the intersection $H \cap BhB$ contains $B_H h B_H$ for all $h \in H$. Now Proposition 4.11 applies to the real reductive group H without modification, whence

$$\nu_H\big(H \cap Be^X B\big) \geq c e^{\rho_{\mathfrak{h}}(X)}$$

for all $X \in \mathfrak{a}_{\mathfrak{h}}$. By the continuity of the homogeneous functions $\rho_{\mathfrak{h}}, \rho_{\mathfrak{g}}$ on $\mathfrak{a}_{\mathfrak{h}}$, there exists a nonzero vector $X \in \mathfrak{a}_{\mathfrak{h}}$ satisfying $\beta_{G/H}\, \rho_{\mathfrak{g}}(X) = \rho_{\mathfrak{h}}(X)$. Setting $g_n := e^{nX} \in H$ with $g_n \to \infty$ in G, we obtain

$$\delta_{G/H} \geq \limsup_{n\to\infty} \frac{\log \nu_H\big(H \cap Be^{nX}B\big)}{\log \nu_G(Be^{nX}B)} \geq \limsup_{n\to\infty} \frac{\log\big(ce^{n\rho_{\mathfrak{h}}(X)}\big)}{\log\big(Ce^{n\rho_{\mathfrak{g}}(X)}\big)} = \beta_{G/H}$$

by applying the lower bound of Proposition 4.11 to H and the upper bound to G. □

4.4.2 Discrete subgroups. For a discrete subgroup Γ, recall that Quint's growth indicator function $\psi_\Gamma : \mathfrak{a}^+ \to \mathbb{R} \cup \{-\infty\}$ is defined by

$$\psi_\Gamma(X) := |X| \inf_{\mathcal{C} \ni X} \inf \left\{ t \in \mathbb{R} : \sum_{\gamma \in \Gamma,\, \kappa(\gamma) \in \mathcal{C}} e^{-t|\kappa(\gamma)|} < \infty \right\},$$

where $\mathcal{C}$ ranges over open cones in $\mathfrak{a}$ which contain X. One has (cf. [27, §I.1])

$$\sup_{\mathfrak{a}^+} \frac{\psi_\Gamma}{2\rho} = \inf \left\{ t \in \mathbb{R} : \sum_{\gamma \in \Gamma} e^{-2t\rho\kappa(\gamma)} < \infty \right\}.$$

We have the following immediate consequence of Proposition 4.14.

Proposition 4.17. *If Γ is a discrete subgroup of G, then*

$$\delta_{G/\Gamma} = \max\left\{ \sup_{X \in \mathfrak{a}^+} \frac{\psi_\Gamma(X)}{2\rho(X)},\, 0 \right\}. \qquad \square$$

Example 4.18. When Γ is a discrete subgroup of $G = \mathrm{SL}(2, \mathbb{R})$, the exponent $\delta_{G/\Gamma}$ coincides with the usual critical exponent δ_Γ. In general, when the semisimple group G is of real rank one, these two exponents are related by the equality $\delta_\Gamma = d_G \delta_{G/\Gamma}$ where the constant d_G depends only on G.

4.4.3 Volume exponents below 1/2. In Section 6 infra we will prove Proposition E which implies that for a closed connected subgroup H, if $\delta_{G/H} > 1/2$, then the two volume exponents $\delta_{G/H} = \beta_{G/H}$ are equal. The following example provides a counterexample to this equality below $1/2$.

Example 4.19. Let $G = \mathrm{SL}(2,\mathbb{R})$ and $N = \left\{ n_s = \begin{pmatrix} 1 & s \\ & 1 \end{pmatrix} : s \in \mathbb{R} \right\}$ be a unipotent subgroup. Then clearly $\beta_{G/N} = 0$.

Yet we claim $\delta_{G/N} = 1/2$. Indeed, computing the singular values gives

$$e^{2\rho\kappa(n_s)} = \frac{s^2 + 2 + \sqrt{s^4 + 4s^2}}{2}, \quad \forall s \in \mathbb{R}.$$

Thus we can compute the relative growth exponent $\delta_{G/N} = 1/2$ through the "Poincaré integral" (Proposition 4.14):

$$I(t) = \int_{\mathbb{R}} \left(\frac{2}{s^2 + 2 + \sqrt{s^4 + 4s^2}} \right)^t \mathrm{d}s < \infty \iff t > 1/2.$$

5 Decay of coefficients and volume growth

The goal of this section is to prove Theorem A on the following relationship

$$\theta_{G/H} = \delta_{G/H} = 1 - \frac{1}{p_{G/H}} \geq \beta_{G/H}$$

between the four exponents we have defined.

Since uniform decay is a strong property, the upper bounds on the other exponents by the coefficient decay exponent $\theta_{G/H}$ are less complicated. The main difficulty, which we will start with, is to establish uniform decay estimates from other data, for which our method is inspired by [24].

Let G be a real semisimple algebraic group and H be a closed subgroup of G. Then by Lemma 2.1, the homogeneous space G/H admits a G-quasi-invariant Radon measure $\mathrm{d}\mu(gH)$ and a continuous density function $\delta : G \to \mathbb{R}^+$ which satisfy (2.2) and (2.3) in Lemma 2.1.

5.1 Matrix coefficients of induced representations

As a preliminary step, we transform the matrix coefficients of $\mathrm{L}^2(G/H)$ into more accessible terms. For later applications as well, we address more generally the coefficients of an induced unitary representation $(\pi, \mathcal{H}) = \mathrm{Ind}_H^G(\sigma, \mathcal{V})$. Setting $\sigma = 1_H$ will recover $\mathrm{L}^2(G/H)$.

Given functions $\varphi_1, \varphi_2 \in C_c(G)$ and vectors $v_1, v_2 \in \mathcal{V}$, consider the elements $f_i := \mathrm{I}(\varphi_i, v_i) \in \mathcal{H}$ for $i = 1, 2$ (constructed in (3.1)). To study

the matrix coefficient $\langle \pi(\cdot) f_1, f_2 \rangle$, we first expand it into integrals on groups by using the expressions of f_1, f_2. We deduce

$$
\begin{aligned}
&\langle \pi(g) f_1, f_2 \rangle \\
&= \int_{G/H} \big\langle f_1(g^{-1}x), f_2(x) \big\rangle_{\mathcal{V}} \left(\frac{\delta(g^{-1}x)}{\delta(x)} \right)^{\frac{1}{2}} \mathrm{d}\mu(xH) && \text{(def of } \mathcal{H}) \\
&= \int_{G/H} \int_H \big\langle \varphi_1(g^{-1}xh)\sigma(h)v_1, f_2(x) \big\rangle_{\mathcal{V}} \left(\frac{\delta(g^{-1}x)}{\delta(x)} \right)^{\frac{1}{2}} \mathrm{d}h \, \mathrm{d}\mu(xH) && \text{(def of } f_1) \\
&= \int_{G/H} \int_H \langle v_1, f_2(xh) \rangle_{\mathcal{V}} \, \varphi_1(g^{-1}xh) \left(\frac{\delta(g^{-1}xh)}{\delta(xh)} \right)^{\frac{1}{2}} \mathrm{d}h \, \mathrm{d}\mu(xH) && (\sigma\text{-eqv}) \\
&= \int_G \langle v_1, f_2(x) \rangle_{\mathcal{V}} \, \varphi_1(g^{-1}x) \big(\delta(g^{-1}x)\delta(x) \big)^{\frac{1}{2}} \mathrm{d}x && \text{(by (2.3))} \\
&= \int_G \int_H \langle v_1, \sigma(h)v_2 \rangle_{\mathcal{V}} \, \varphi_1(g^{-1}x) \overline{\varphi_2(xh)} \, \big(\delta(g^{-1}x)\delta(x) \big)^{\frac{1}{2}} \mathrm{d}h \, \mathrm{d}x && \text{(def of } f_2) \\
&= \int_H \langle v_1, \sigma(h)v_2 \rangle_{\mathcal{V}} \int_G \varphi_1(g^{-1}x) \overline{\varphi_2(xh)} \, \big(\delta(g^{-1}x)\delta(x) \big)^{\frac{1}{2}} \mathrm{d}x \, \mathrm{d}h && \text{(Fubini)} \\
&= \int_H \langle v_1, \sigma(h)v_2 \rangle_{\mathcal{V}} \int_G \varphi_1(g^{-1}x) \overline{\varphi_2(xh)} \, \big(\delta(g^{-1}x)\delta(xh) \big)^{\frac{1}{2}} \mathrm{d}x \, \mathrm{d}\nu_H(h), && \text{(prop of } \delta)
\end{aligned}
$$

where we recall that $\mathrm{d}\nu_H(h) = \Delta_H(h)^{-1/2} \, \mathrm{d}h$ is the symmetric measure on H. By changing h to h^{-1}, we obtain

$$
\langle \pi(g) f_1, f_2 \rangle = \int_H \langle \sigma(h)v_1, v_2 \rangle_{\mathcal{V}} \, \Phi(h, g) \, \mathrm{d}\nu_H(h), \tag{5.1}
$$

where we define

$$
\Phi(h, g) := \int_G \varphi_1(g^{-1}x) \overline{\varphi_2(xh^{-1})} \, \big(\delta(g^{-1}x)\delta(xh^{-1}) \big)^{\frac{1}{2}} \mathrm{d}x. \tag{5.2}
$$

Using the volume decay in G studied in the last section, we can now obtain the first estimates for decay of coefficients.

Lemma 5.1. *Let B_G be the open neighborhood of $e \in G$ given in Lemma 4.2. For any $v_1, v_2 \in \mathcal{V}$ and any $\varphi_1, \varphi_2 \in C_{\mathrm{c}}(G)$ whose supports $B_i := \operatorname{supp} \varphi_i$ satisfy $B_i B_i^{-1} \subset B_G$, let $f_i := \mathrm{I}(\varphi_i, v_i) \in \mathcal{H}$ for $i = 1, 2$. Then there exists a constant $C > 0$ such that we have uniformly for all $g \in G$,*

$$
|\langle \pi(g) f_1, f_2 \rangle| \leq C e^{-2\rho\kappa(g)} \int_{H \cap (B_2^{-1} g B_1)} |\langle \sigma(h)v_1, v_2 \rangle_{\mathcal{V}}| \, \mathrm{d}\nu_H(h).
$$

Proof. If $\Phi(h, g) \neq 0$, then there exists $x_0 \in G$ such that $g^{-1}x_0 =: b_1 \in B_1$ and $x_0 h^{-1} =: b_2 \in B_2$, so $h = b_2^{-1} g b_1 \in B_2^{-1} g B_1$. By using the unimodular-

ity of G, we deduce

$$\begin{aligned}\Phi(h,g) &= \int_G \varphi_1(g^{-1}x)\overline{\varphi_2(xb_1^{-1}g^{-1}b_2)}\left(\delta(g^{-1}x)\delta(xh^{-1})\right)^{\frac{1}{2}}\mathrm{d}x \\ &= \int_G \varphi_1(g^{-1}ygb_1)\overline{\varphi_2(yb_2)}\left(\delta(g^{-1}ygb_1)\delta(yb_2)\right)^{\frac{1}{2}}\mathrm{d}y \qquad (5.3)\end{aligned}$$

by setting $y = xb_1^{-1}g^{-1} = xh^{-1}b_2^{-1}$. For $i = 1, 2$, define

$$\tilde{\varphi}_i(x) := \sup_{b\in B_i} |\varphi_i(xb)|\,\delta(xb)^{\frac{1}{2}}.$$

Then $\tilde{\varphi}_i \in C_{\mathrm{c}}(B_G)$ by the hypothesis on B_i. Applied to $\tilde{\varphi}_1, \tilde{\varphi}_2$, Proposition 4.3 yields

$$\int_G \tilde{\varphi}_1(g^{-1}yg)\tilde{\varphi}_2(y)\,\mathrm{d}y \leq Ce^{-2\rho\kappa(g)}$$

uniformly for all $g \in G$. Feeding back to (5.3), we deduce

$$|\Phi(h,g)| \leq Ce^{-2\rho\kappa(g)}\mathbb{1}_{B_2^{-1}gB_1}(h)$$

uniformly for $g \in G$ and $h \in H$. We conclude by feeding back to (5.1). □

5.2 From volume growth to uniform decay

Proposition 5.2. $\theta_{G/H} \leq \delta_{G/H}$.

Proof. By Corollary 3.3, the subset

$$\left\{\mathrm{I}(\varphi, 1) : \varphi \in C_{\mathrm{c}}(G) \text{ with } (\operatorname{supp}\varphi)(\operatorname{supp}\varphi)^{-1} \subset B_G\right\}$$

spans the space $C_{\mathrm{c}}(G/H)$. By applying Lemma 5.1 to the induced representation $\lambda_{G/H} = \operatorname{Ind}_H^G 1_H$, we deduce for any $f_1, f_2 \in C_{\mathrm{c}}(G/H)$ that

$$\left|\left\langle\lambda_{G/H}(g)f_1, f_2\right\rangle\right| \leq C_1e^{-2\rho\kappa(g)}\nu_H\left(H \cap B_2^{-1}gB_1\right)$$

uniformly for $g \in G$, for some compacta B_1, B_2 of G. Choose $B \Subset G$ which contains $B_2^{-1} \cup B_1$. By the definition of $\delta_{G/H}$ and by Proposition 4.11, given any number $\delta > \delta_{G/H}$, there exists a constant $C_1 > 0$ such that

$$\nu_H\left(H \cap B_2^{-1}gB_1\right) \leq \nu_H(H \cap BgB) \leq C_2e^{2\delta\rho\kappa(g)}$$

for all $g \in G$, whence

$$\left|\left\langle\lambda_{G/H}(g)f_1, f_2\right\rangle\right| \leq C_3e^{-2(1-\delta)\rho\kappa(g)}$$

uniformly for $g \in G$. Since this is valid for any functions $f_1, f_2 \in C_{\mathrm{c}}(G/H)$ and any number $\delta > \delta_{G/H}$, we have $\delta_{G/H} \geq \theta_{G/H}$. □

5.3 From uniform decay to integrability

Recall that the integrability exponent $p_{G/H}$ optimizes the condition that for all $f_1, f_2 \in C_c(G/H)$ (or equivalently in $L_c^\infty(G/H)$), we have

$$\langle \lambda_{G/H}(\cdot) f_1, f_2 \rangle \in \bigcap_{p > p_{G/H}} L^p(G).$$

Proposition 5.3. $1 - 1/p_{G/H} \leq \theta_{G/H}$.

Proof. Suppose $\theta_{G/H} < 1$. For any $\theta > \theta_{G/H}$ and any $f_1, f_2 \in C_c(G/H)$, we obtain from Definition 1.1 that

$$\left| \langle \lambda_{G/H}(g) f_1, f_2 \rangle \right| \leq C e^{(2\theta - 2)\rho\kappa(g)}$$

uniformly for $g \in G$. Then by applying Lemma 2.7, we have

$$\int_G \left| \langle \lambda_{G/H}(g) f_1, f_2 \rangle \right|^p \mathrm{d}g \leq C \int_{\mathfrak{a}^+} e^{(2\theta p - 2p + 2)\rho(X)} \, \mathrm{d}X,$$

which is finite as long as $\theta < 1 - 1/p$. By the arbitrariness of $\theta > \theta_{G/H}$, we conclude that $1 - 1/p_{G/H} \leq \theta_{G/H}$. □

5.4 From integrability to volume growth

Lemma 5.4. *Let B be a symmetric compact neighborhood of e in G. For*

$$f(xH) = \int_H \mathbb{1}_B(xh) \delta(xh)^{-\frac{1}{2}} \, \mathrm{d}h \in L_c^\infty(G/H),$$

we have uniformly for all $g \in G$ that

$$\int_{BBgBB} \langle \lambda_{G/H}(x) f, f \rangle \, \mathrm{d}x \geq \nu_G(B)^2 \nu_H(H \cap BgB),$$

where $\mathrm{d}x$ denotes the Haar measure of G.

Proof. Let us apply the preliminary computations in Section 5.1 to the induced representation $\lambda_{G/H} = \mathrm{Ind}_H^G 1_H$ and the functions

$$\varphi_1(x) = \varphi_2(x) = \mathbb{1}_B(x) \delta(x)^{-\frac{1}{2}} \in L_c^\infty(G),$$

so that the functions f_1, f_2 therein coincide with f. Now (5.2) becomes $\Phi(h, g) = \nu_G(gB \cap Bh)$ and thus (5.1) yields

$$\langle \lambda_{G/H}(g) f_1, f_2 \rangle = \int_H \nu_G(gB \cap Bh) \, \mathrm{d}\nu_H(h).$$

Let $B_g := BBgBB$ for $g \in G$. By Fubini's theorem, we have

$$\begin{aligned}\int_{B_g} \langle \lambda_{G/H}(x) f_1, f_2 \rangle \, \mathrm{d}x &= \int_{B_g} \int_H \nu_G(xB \cap Bh) \, \mathrm{d}\nu_H(h) \, \mathrm{d}x \\ &\geq \int_{H \cap BgB} \int_G \mathbb{1}_{B_g}(x) \nu_G(xB \cap Bh) \, \mathrm{d}x \, \mathrm{d}\nu_H(h). \end{aligned} \tag{5.4}$$

Given any $h \in H \cap BgB$, we can write $h = b_1 g b_2$ for some $b_1, b_2 \in B$. Then from Fubini's theorem, the symmetric property of the subset B, and the unimodularity of the group G, we deduce

$$\begin{aligned}\int_G \mathbb{1}_{B_g}(x) \nu_G(xB \cap Bh) \, \mathrm{d}x &= \int_G \int_G \mathbb{1}_{B_g}(x) \mathbb{1}_{xB}(y) \mathbb{1}_{Bb_1gb_2}(y) \, \mathrm{d}y \, \mathrm{d}x \\ &= \int_G \nu_G(BBgBB \cap yB) \mathbb{1}_{Bb_1gb_2}(y) \, \mathrm{d}y \\ &\geq \int_{Bb_1gb_2} \nu_G(B) \, \mathrm{d}y = \nu_G(B)^2, \end{aligned}$$

whence we conclude by feeding back to (5.4). □

Proposition 5.5. $\delta_{G/H} \leq 1 - 1/p_{G/H}$.

Proof. By the definition of the relative volume growth exponent $\delta_{G/H}$ and Remark 4.13, for any number $\delta < \delta_{G/H}$ there exists a compact, symmetric, K-bi-invariant neighborhood B of e in G, and a sequence (X_n) in $\mathfrak{a}^+$ going to infinity, such that

$$\log \nu_H\big(H \cap Be^{X_n}B\big) \geq \delta \log \nu_G\big(Be^{X_n}B\big)$$

uniformly for $n \in \mathbb{N}$, whence we can deduce from Proposition 4.11 that

$$\nu_H\big(H \cap Be^{X_n}B\big) \geq c_1 e^{2\delta\rho(X_n)} \tag{5.5}$$

uniformly for $n \in \mathbb{N}$. By passing to a subsequence, we can assume that the subsets $BBe^{X_n}BB$ with $n \in \mathbb{N}$ are pairwise disjoint.

Let the function $f \in \mathrm{L}^2(G/H)$ be given as in Lemma 5.4. We proceed to study the integrability of the matrix coefficient $\langle \lambda_{G/H}(\cdot) f, f \rangle$. By applying the Hölder inequality, we obtain for $p > 1$ that

$$\int_{BBgBB} \big|\langle \lambda_{G/H}(x) f, f \rangle\big|^p \, \mathrm{d}x \geq \nu_G(BBgBB)^{1-p} \left(\int_{BBgBB} \langle \lambda_{G/H}(x) f, f \rangle \, \mathrm{d}x \right)^p.$$

Further from Lemma 5.4 and Proposition 4.11, we deduce

$$\int_{BBgBB} \big|\langle \lambda_{G/H}(x) f, f \rangle\big|^p \, \mathrm{d}x \geq c_2 e^{2(1-p)\rho\kappa(g)} \nu_H(H \cap BgB)^p$$

uniformly for all $g \in G$. By setting $g = e^{X_n}$ and applying (5.5), we obtain

$$\int_{BBe^{X_n}BB} \left|\left\langle \lambda_{G/H}(x)f, f\right\rangle\right|^p \mathrm{d}x \geq c_3 e^{(2-2p+2p\delta)\rho(X_n)}$$

uniformly for $n \in \mathbb{N}$. If $\left\langle \lambda_{G/H}(\cdot)f, f\right\rangle \in \mathrm{L}^p(G)$, then we have

$$\sum_{n\in\mathbb{N}} e^{2(1-p+p\delta)\rho(X_n)} < \infty,$$

which implies that $1 - 1/p > \delta$. By the arbitrariness of $p > p_{G/H}$ and $\delta < \delta_{G/H}$, we conclude that $1 - 1/p_{G/H} \geq \delta_{G/H}$. □

5.5 Local lower bound of coefficient decay

Proposition 5.6. *For any closed subgroup H of G, we have*

$$\beta_{G/H} \leq \theta_{G/H}.$$

Proof. The philosophy of the proof is that the local dynamics near $eH \in G/H$ mimics the dynamics on $\mathfrak{g}/\mathfrak{h}$. We shall incarnate this idea, even though generally we have no canonical local chart for the homogeneous space G/H. Let us decompose the proof into several steps.

Step 1. Preliminary reduction.

It suffices to show that $\rho_{\mathfrak{h}}(X) \leq \theta_{G/H}\rho_{\mathfrak{g}}(X)$ for every element $X \in \mathfrak{h}$. From now on we fix a nonzero $X \in \mathfrak{h}$ which, by the Jordan decomposition, splits uniquely as a commuting sum $X = X_s + X_n$ of a semisimple element X_s and a nilpotent element X_n inside $\mathfrak{g}$, meaning that the operators $\operatorname{ad} X_s, \operatorname{ad} X_n \in \mathrm{GL}(\mathfrak{g})$ are respectively semisimple, nilpotent. In general, the element X_s may not be in $\mathfrak{h}$, but we always have $[X_s, \mathfrak{h}] \subset \mathfrak{h}$ since $\operatorname{ad} X_s$ can be written as a real polynomial of $\operatorname{ad} X$. We have $\rho_{\mathfrak{h}}(\operatorname{ad} X_s) = \rho_{\mathfrak{h}}(X)$ and $\rho_{\mathfrak{g}}(X_s) = \rho_{\mathfrak{g}}(X)$. Write $x_s := \operatorname{ad} X_s$ and $x_n := \operatorname{ad} X_n$.

We henceforth fix a norm $|\cdot|$ on $\mathfrak{g}$. By the nilpotency of $x_n \in \mathrm{End}(\mathfrak{g})$, for some constant $C \geq 1$ and $d = \dim \mathfrak{g}$ we have

$$\|\exp(-tx_n)\|_{\mathrm{op}} \leq Ct^d, \quad \text{for all } t \geq 1. \tag{5.6}$$

Step 2. A choice of the local chart.

By the semisimplicity of x_s, there is an x_s-invariant complementary subspace $\mathfrak{w}$ with $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{w}$, but $\mathfrak{w}$ is not necessarily invariant by $\operatorname{ad} X$ or x_n. Now the space $\mathfrak{w}$ is naturally identified with $\mathfrak{g}/\mathfrak{h}$, with the same action of x_s, and we will use the local chart $E : \mathfrak{w} \to G/H$ with $E(Y) := e^Y H$. We fix a convex compact neighborhood D of 0 in $\mathfrak{w}$ such that E restricts a homeomorphism onto its image denoted by $B \Subset G/H$. Note that

$$\text{there exists } r > 0, \text{ such that } e^Y H \in B \text{ for all } Y \in \mathfrak{g} \text{ with } |Y| \leq r. \tag{5.7}$$

Up to shrinking D, we can assume that the elements in D have norm less than r and we have $E(D) \subset B$ instead of equality.

Step 3. A technical lemma.

Lemma 5.7. *For all $t \geq 1$, let $D_t := \frac{1}{Ct^d} D \Subset \mathfrak{w}$. Then we have*

$$E(\exp(tx_s)D_t \cap D_t) \subset e^{tX} B \cap B.$$

Proof (lemma). Let $Y \in \exp(tx_s)D_t \cap D_t$. Then we have $e^Y H \in B$ since $D_t \subset D$. It suffices to prove that $e^{-tX} e^Y H \in B$. By assumption, there exists $Y_0 \in D_t$ with $Y = e^{tx_s} Y_0$. We deduce $\mathrm{Ad}(e^{-tX})Y = e^{-tx_n} Y_0$ and furthermore by (5.6) that

$$\left|\mathrm{Ad}(e^{-tX})Y\right| \leq \left\|\exp(-tx_n)\right\|_{\mathrm{op}} |Y_0| \leq r.$$

By (5.7) and the fact that $X \in \mathfrak{h}$, we have

$$e^{\mathrm{Ad}(e^{-tX})Y} H = e^{-tX} e^Y H \in B. \qquad \square$$

Step 4. Local decay from linear algebra.

By Lemma 5.7 and by taking into account the Radon–Nikodym derivatives, we deduce, uniformly for $t \geq 1$ that

$$\left\langle \lambda_{G/H}(e^{tX}) \mathbb{1}_B, \mathbb{1}_B \right\rangle \geq c_1 \, e^{-\operatorname{tr}_{\mathfrak{w}}(tx_s)/2} \operatorname{Vol}_{\mathfrak{w}}\left(e^{tx_s} D_t \cap D_t\right),$$

with $D_t = \frac{1}{Ct^d} D$. By Corollary 4.8, for the constant $d_1 := d \dim \mathfrak{w}$ and for all $t \geq 1$ we have uniformly,

$$e^{-\operatorname{tr}_{\mathfrak{w}}(tx_s)/2} \operatorname{Vol}_{\mathfrak{w}}\left(e^{tx_s} D_t \cap D_t\right) \geq c_2 t^{-d_1} e^{-t\rho_{\mathfrak{w}}(x_s)}.$$

But since $\rho_{\mathfrak{w}}(x_s) = \rho_{\mathfrak{w}}(X)$ (by $[x_s, x_n] = 0$), we have for all $t \geq 1$,

$$\left\langle \lambda_{G/H}(e^{tX}) \mathbb{1}_B, \mathbb{1}_B \right\rangle \geq c_3 t^{-d_1} e^{-t\rho_{\mathfrak{w}}(X)}.$$

But by the definition of $\theta_{G/H}$, for any $\theta > \theta_{G/H}$ we have uniformly for $t > 0$,

$$\left\langle \lambda_{G/H}(e^{tX}) \mathbb{1}_B, \mathbb{1}_B \right\rangle \leq C_4 e^{(\theta-1)\rho_{\mathfrak{g}}(tX)}.$$

By sending $t \to +\infty$ in $t^{-d_1} e^{-t\rho_{\mathfrak{w}}(X)} \leq C_5 e^{(\theta-1)t\rho_{\mathfrak{g}}(X)}$, we get

$$-\rho_{\mathfrak{g}}(X) + \rho_{\mathfrak{h}}(X) = \rho_{\mathfrak{w}}(X) \leq (\theta - 1)\rho_{\mathfrak{g}}(X),$$

whence $\rho_{\mathfrak{h}}(X) \leq \theta_{G/H} \rho_{\mathfrak{g}}(X)$, completing the proof of Proposition 5.6. $\square$

5.6 Conclusion of proofs

Proof of Theorem A. The combination of Proposition 5.2, Proposition 5.3, Proposition 5.5, and Proposition 5.6 yields

$$\delta_{G/H} \geq \theta_{G/H} \geq 1 - \frac{1}{p_{G/H}} \geq \delta_{G/H}, \quad \theta_{G/H} \geq \beta_{G/H}.$$

Hence, we conclude the proof of the theorem. □

Applications to special classes of subgroups are immediate.

Proof of Corollary C. When the subgroup H is reductive, it follows from Theorem A and Proposition 4.15 that

$$\theta_{G/H} = \delta_{G/H} = \beta_{G/H}.$$ □

Proof of Corollary D. When the subgroup Γ is discrete, it follows from Theorem A and Proposition 4.17 that

$$\theta_{G/\Gamma} = \delta_{G/\Gamma} = \max\left\{\sup_{\mathfrak{a}^+} \frac{\psi_\Gamma}{2\rho}, 0\right\}.$$ □

6 Uniform decay of induced representations

The goal of this section is to prove Proposition E. Thanks to Theorem A, it will suffice to show that above $1/2$, the coefficient decay exponent $\theta_{G/H}$ is bounded from above by the local volume decay exponent $\beta_{G/H}$.

We will first prove the statement for algebraic subgroups (Sections 6.1–6.3) and then extend to all closed subgroups with finitely many components (Section 6.4). We will establish uniform decay estimates by applying the strategy of Benoist–Kobayashi in [5]. More specifically, through a chain of algebraic subgroups, we will reduce the unitary induction from H to G into the composition of several ones which are easier to handle. The rich structure theory of semisimple groups allows us to absorb the local data into each intermediate induced representation.

6.1 Ingredients from Benoist–Kobayashi

From now on, we fix H to be an algebraic subgroup of the real semisimple algebraic group G. Since the values of the four exponents remain unchanged if we pass to an open subgroup of finite index, we can assume H to be Zariski connected.

In what follows, we recall several ingredients in the strategy of Benoist–Kobayashi in [5]. The first ingredient is the existence of nice intermediate subgroups.

Lemma 6.1 ([5, Lem 4.1]). *There exist two intermediate algebraic subgroups $H \subset F \subset Q \subset G$ with the following properties:*

(1) Q is a parabolic subgroup of G of minimal dimension containing H.

(2) Let U be the unipotent radical of Q. There exists a Levi decomposition $Q = LU$ such that L is a maximal reductive subgroup of Q and that $H = (L \cap H)(U \cap H)$.

(3) $S := L \cap H$ is a maximal reductive subgroup of H and $V := U \cap H$ is the unipotent radical of H.

(4) $F = SU$.

Thus, we have a chain of algebraic subgroups with compatible Levi decompositions

$$H = SV \subset F = SU \subset Q = LU \subset G. \qquad \square$$

The notation of these groups will be standing from Section 6.1 to Section 6.3, and their Lie algebras will be denoted by the corresponding Fraktur letters. We can suppose that $Q \neq G$, for otherwise the algebraic subgroup H is already reductive and we can conclude by Corollary C.

The second ingredient is the domination of group actions. On the homogeneous space U/V, the reductive group S acts by conjugation and the unipotent group U acts by left translation.

Lemma 6.2 ([5, Prop 4.4]). *Let U/V be equipped with a U-invariant Radon measure* Vol. *Then for every compact subset $D \Subset U/V$, there exists a compact subset $D_0 \Subset U/V$ such that we have for all $s \in S$ and $u \in U$,*

$$\mathrm{Vol}(suD \cap D) \leq \mathrm{Vol}(sD_0 \cap D_0). \qquad \square$$

Since $F/H = U/V$, this lemma states that sufficiently many matrix coefficients of the unitary representation $(\lambda_{F/H}, \mathrm{L}^2(F/H))$ are dominated by those of the unitary representation $(\sigma_0, \mathrm{L}^2(U/V))$ of F, whose action is given by $\sigma_0(su) = \lambda_{F/H}(s)$ for all $s \in S$ and $u \in U$, cf. [6, §4.1].

Now this majoration carries to induced representations (cf. [6, §4.2]), which implies that the coefficients of $\lambda_{Q/H} = \mathrm{Ind}_F^Q \lambda_{F/H}$ (Lemma 3.4) are dominated by those of the induced representation $\pi_0 := \mathrm{Ind}_F^Q \sigma_0$. We have thus obtained the following conclusion, cf. also [5, Prop 4.9].

Lemma 6.3. *For any $f_1, f_2 \in C_c(Q/H)$, there exist a nonnegative function $\varphi_0 \in C_c(L)$ and a compact subset $D_0 \Subset U/V$ such that we have uniformly for all $q \in Q$,*

$$\left|\left\langle \lambda_{Q/H}(q) f_1, f_2 \right\rangle\right| \leq \left\langle \pi_0(l_q) \mathrm{I}_S^L(\varphi_0, \mathbb{1}_{D_0}), \mathrm{I}_S^L(\varphi_0, \mathbb{1}_{D_0}) \right\rangle,$$

where l_q is the L-component of q in $Q = LU$. $\square$

Note that the right hand side depends only on the restricted representation $\pi_0|_L$. We claim that $\pi_0|_L \cong \mathrm{Ind}_S^L(\sigma_0|_S)$. Indeed, since the unitary representation σ_0 is trivial on the unipotent radical U, the induced representation π_0 is also trivial on U.

The last ingredient is more classical. The reductive group S acts by conjugation on the pair of real unipotent Lie groups $U \supset V$. By the theory of adapted bases for nilpotent Lie algebras, there exists an S-invariant complementary space $\mathfrak{u} = \mathfrak{w} \oplus \mathfrak{v}$ such that the exponential map induces an S-equivariant homeomorphism $\exp : \mathfrak{w} \to U/V$, cf. [6, Lem 4.7]. Hence, the representation $(\sigma_0|_S, \mathrm{L}^2(U/V))$ is equivalent to the unitary representation $(\tau, \mathrm{L}^2(\mathfrak{u}/\mathfrak{v}))$ of S derived from the linear representation of S on the quotient space $\mathfrak{u}/\mathfrak{v} \cong \mathfrak{w}$, as in (4.1). Then Corollary 4.9 implies immediately the following estimates.

Corollary 6.4. *Fix a Cartan decomposition $S = K_S A_S K_S$ with the Cartan projection $\kappa_S : S \to \mathfrak{a}_\mathfrak{s}/W_S$. For any compact subset $D \Subset U/V$, there exists a constant $C > 0$ such that we have uniformly for all $s \in S$,*

$$|\langle \sigma_0(s)\mathbb{1}_D, \mathbb{1}_D\rangle| \leq C \exp\big\{-\rho_{\mathfrak{u}/\mathfrak{v}}\kappa_S(s)\big\}. \qquad \square$$

6.2 Uniform decay of reductive induction

Next, we inspect the uniform decay of $\pi_0|_L = \mathrm{Ind}_S^L(\sigma_0|_S)$.

First recall some notation from Section 2.2.2. The real reductive group L admits the maximal compact subgroup K_M which is contained in K and the Cartan subspace A which is also a Cartan subspace in G. The Cartan projection of L, denoted by $\kappa_L : L \to \mathfrak{a}$, is determined by the positive system Σ_M^+ and $\rho_L = \rho_M$. The Weyl group W_L acting on $\mathfrak{a}$ is thus identified with a subgroup of W_G.

The following decay estimates bring the local volume decay exponent $\beta_{G/H}$ into play. The proof relies on a refined control of the preliminary decay estimates in Section 5.1.

Lemma 6.5. *Given any function $\varphi \in C_\mathrm{c}(L)$ and compactum $D \Subset U/V$, we form the vector $f = \mathrm{I}_S^L(\varphi, \mathbb{1}_D)$ in the representation space of $\pi_0|_L$ identified with $\mathrm{Ind}_S^L(\sigma_0|_S)$. Then there exists a constant $C > 0$ such that we have uniformly for all $l \in L$,*

$$|\langle \pi_0(l)f, f\rangle| \leq C \exp\left\{\frac{1}{2}\big(-\rho_\mathfrak{l} + (2\beta_{G/H} - 1)\rho_\mathfrak{g}\big)(\kappa_L(l))\right\}.$$

Proof. By applying Lemma 5.1 to $\pi_0|_L = \mathrm{Ind}_S^L(\sigma_0|_S)$ and Lemma 3.2 to φ, we have, for some compactum $B \Subset L$, uniformly for all $l \in L$ that

$$|\langle \pi_0(l)f, f\rangle| \leq C_1 \exp\{-\rho_\mathfrak{l}\kappa_L(g)\} \int_{S\cap BlB} |\langle \sigma_0(s)\mathbb{1}_D, \mathbb{1}_D\rangle| \, \mathrm{d}s. \tag{6.1}$$

Next, we inspect the integral term. Since the Cartan subspace $\mathfrak{a}_{\mathfrak{s}}$ of S lies in some Cartan subspace of L which is conjugate by L to the subspace $\mathfrak{a}$, we can identify $\mathfrak{a}_{\mathfrak{s}}$ with a subspace of $\mathfrak{a}$. Write $X = \kappa_L(l) \in \mathfrak{a}$. As in (4.4) in the proof of Lemma 4.16, we have for some uniform constant $r > 0$ that

$$S \cap BlB \subset \bigcup_{w \in W_L} K_S \exp\{(wX + \mathfrak{a}(r)) \cap \mathfrak{a}_{\mathfrak{s}}\} K_S.$$

Combining this with Corollary 6.4 yields the following uniform estimates

$$\begin{aligned}
&\int_{S\cap BlB} |\langle \sigma_0(s)\mathbb{1}_D, \mathbb{1}_D\rangle| \, \mathrm{d}s \leq C_2 \int_{S\cap BlB} \exp\big\{-\rho_{\mathfrak{u}/\mathfrak{v}}\kappa_S(s)\big\} \, \mathrm{d}s \\
&\leq C_2 \sum_{w\in W_L} \int_{K_S \exp\{(wX+\mathfrak{a}(r))\cap\mathfrak{a}_{\mathfrak{s}}\}K_S} \exp\big\{-\rho_{\mathfrak{u}/\mathfrak{v}}\kappa_S(s)\big\} \, \mathrm{d}s \\
&\leq C_3 \sum_{w\in W_L} \int_{(wX+\mathfrak{a}(r))\cap\mathfrak{a}_{\mathfrak{s}}} \exp\big\{\rho_{\mathfrak{s}}(Y) - \rho_{\mathfrak{u}/\mathfrak{v}}(Y)\big\} \, \mathrm{d}Y, \qquad (6.2)
\end{aligned}$$

where the last inequality follows from Lemma 2.7. But since one has the following equalities and inequality of the rho-functions on $\mathfrak{a}_{\mathfrak{s}}$.

$$\rho_{\mathfrak{s}} - \rho_{\mathfrak{u}/\mathfrak{v}} = \rho_{\mathfrak{s}} + \rho_{\mathfrak{v}} - \rho_{\mathfrak{u}} = \rho_{\mathfrak{h}} - (\rho_{\mathfrak{g}} - \rho_{\mathfrak{l}})/2 \leq \big(\beta_{G/H} - 1/2\big)\rho_{\mathfrak{g}} + \rho_{\mathfrak{l}}/2,$$

we deduce from the W_L-invariance and the uniform Lipschitz property of the rho-functions $\rho_{\mathfrak{g}}, \rho_{\mathfrak{l}}$ on $\mathfrak{a}$ that (6.2) is further bounded from above, uniformly for all $l \in L$, by

$$C_4 \exp\left\{\frac{1}{2}\left(\rho_{\mathfrak{l}} + (2\beta_{G/H} - 1)\rho_{\mathfrak{g}}\right)(X)\right\}.$$

We conclude the proof by feeding back to (6.1). □

Recall from Section 2.3 that the spherical functions of the real reductive group $L = K_M A K_M$ are given for $\chi \in \mathfrak{a}'$ by

$$\Xi^L_\chi(l) = \int_{K_M} e^{-(\chi+\rho_L)\eta(l^{-1}k_M)} \, \mathrm{d}k_M.$$

Let us write $t^+ := \max\{t, 0\}$ for $t \in \mathbb{R}$.

Corollary 6.6. *Let $\chi = (2\beta_{G/H} - 1)^+\rho \in \mathfrak{a}'$. Under the same assumptions as Lemma 6.5, we have uniformly for all $l \in L$,*

$$|\langle \pi_0(l)f, f\rangle| \leq C \sum_{w\in W_G} \Xi^L_{w\chi}(l).$$

Proof. If $\beta_{G/H} \leq 1/2$, then Lemma 6.5 implies that $\pi_0|_L$ is tempered. We have $\chi = 0$ and we conclude by Theorem 3.9.

From now on assume $\beta_{G/H} > 1/2$. By Example 4.6, we have the equality $\rho_{\mathfrak{l}} \circ \kappa_L = 2\rho_L \circ \kappa_L$ on L and $\rho_{\mathfrak{g}} = 2\max_{w\in W_G} w\rho$ on $\mathfrak{a}$. Thus, the uniform estimates of Lemma 6.5 yields uniformly for all $l \in L$,

$$|\langle \pi_0(l)f, f\rangle| \leq C \max_{w\in W_G} \exp\big\{\big(-\rho_L + (2\beta_{G/H} - 1)w\rho\big)(\kappa_L(l))\big\}.$$

But then Lemma 2.11 applied to the reductive group L gives the uniform majoration by spherical functions

$$|\langle \pi_0(l)f, f\rangle| \leq C \max_{w\in W_G} \Xi^L_{w\chi}(l) \leq C \sum_{w\in W_G} \Xi^L_{w\chi}(l). \qquad \square$$

To summarize, the domination of $\lambda_{Q/H}$ by π_0 (Lemma 6.3) and the decay estimates of π_0 (Corollary 6.6) together yield the following uniform decay estimates of the quasi-regular representation $\lambda_{Q/H}$.

Corollary 6.7. *Let $\chi_0 = (2\beta_{G/H} - 1)^+\rho$. Then for any $\xi_1, \xi_2 \in C_c(Q/H)$, there exists a constant $C > 0$ such that we have uniformly for all $q \in Q$,*

$$\big|\big\langle \lambda_{Q/H}(q)\xi_1, \xi_2\big\rangle\big| \leq C \sum_{w\in W_G} \Xi^L_{w\chi_0}(l_q),$$

where l_q denotes the L-component of q in $Q = LU$. $\square$

6.3 Parabolic induction and spherical functions

With the premise of Corollary 6.7, the next step of the proof is to establish uniform decay estimates for a unitary representation of G which is induced from the parabolic subgroup Q (Proposition 6.9). This will allow use to control the coefficient decay exponent of algebraic subgroups.

The following lemma is a variant of [19, Prop 7.14], highlighting the utility of spherical functions. One should think of both sides of (6.3) as the matrix coefficients of certain parabolically induced representations.

Lemma 6.8. *Let $B \Subset G$ be a compactum of nonempty interior and let $\chi \in \mathfrak{a}'$ be a real linear form. Then there exists a constant $C = C(B, \chi) > 0$, such that uniformly for all $g \in G$, we have*

$$e^{-2\rho\kappa(g)} \int_{Q\cap BgB} \Xi^L_\chi(l_q^{-1})\, e^{\rho_Q \eta(q)}\, \mathrm{d}q \leq C\, \Xi^G_\chi(g), \tag{6.3}$$

where $\mathrm{d}q$ is the left Haar measure on Q, and l_q denotes the L-component of $q \in Q = LU$.

Proof. Recall the notation from Section 2.2.2. We have $N_Q = U$ as the unipotent radical of Q. Since the group K_M normalizes N_Q and the map $\eta(\cdot)$ is N_Q-right-invariant, for any $q \in Q$ we have

$$\Xi_\chi^L(l_q^{-1}) = \int_{K_M} e^{-(\chi+\rho_L)\eta(l_q k_M)}\,\mathrm{d}k_M = \int_{K_M} e^{-(\chi+\rho_L)\eta(q k_M)}\,\mathrm{d}k_M.$$

The integral term in the LHS of (6.3) is thus bounded from above by

$$\int_Q \int_{K_M} \mathbb{1}_{BgB}(q)\, e^{-(\chi+\rho_L)\eta(qk_M)+\rho_Q\eta(q)}\,\mathrm{d}k_M\,\mathrm{d}q. \tag{6.4}$$

Up to enlarging B, we can assume the compactum B to be K-bi-invariant. Note that the function

$$x \mapsto \int_{K_M} \mathbb{1}_{BgB}(x)\, e^{-(\chi+\rho_L)\eta(xk_M)+\rho_Q\eta(x)}\,\mathrm{d}k_M$$

is K-left-invariant and integrable on G. By the integral formula for the decomposition $G = KQ$ (Lemma 2.6), (6.4) equals

$$\int_{BgB} \int_{K_M} e^{-(\chi+\rho_L)\eta(xk_M)-\rho_Q\eta(x)}\,\mathrm{d}k_M\,\mathrm{d}x. \tag{6.5}$$

But since we have $BgB \subset Ke^{\kappa(g)+\mathfrak{a}(r)}K$ (Lemma 2.3) for some constant $r = r(B)$, the integral formula for the Cartan decomposition (Lemma 2.7) yields that (6.5) is bounded from above by

$$\begin{aligned}
&\int_K \int_{\kappa(g)+\mathfrak{a}(r)} \int_K \int_{K_M} e^{-(\chi+\rho_L)\eta(k_1 e^X k_2 k_M)-\rho_Q\eta(k_1 e^X k_2)+2\rho(X)}\,\mathrm{d}k_M\,\mathrm{d}k_2\,\mathrm{d}X\,\mathrm{d}k_1 \\
&= \int_{\kappa(g)+\mathfrak{a}(r)} \int_K \int_{K_M} e^{-(\chi+\rho_L)\eta(e^X k k_M)-\rho_Q\eta(e^X k)+2\rho(X)}\,\mathrm{d}k_M\,\mathrm{d}k\,\mathrm{d}X \\
&\le C_1 e^{2\rho\kappa(g)} \int_{\kappa(g)+\mathfrak{a}(r)} \int_K \int_{K_M} e^{-(\chi+\rho_L)\eta(e^X k k_M)-\rho_Q\eta(e^X k)}\,\mathrm{d}k_M\,\mathrm{d}k\,\mathrm{d}X,
\end{aligned} \tag{6.6}$$

uniformly for all $g \in G$. But since $\rho_Q\eta(xk_M) = \rho_Q\eta(x)$ for any $k_M \in K_M$ (Lemma 2.5), we deduce from Fubini and $\rho = \rho_Q + \rho_L$ that

$$\begin{aligned}
&\int_K \int_{K_M} e^{-(\chi+\rho_L)\eta(e^X k k_M)-\rho_Q\eta(e^X k)}\,\mathrm{d}k_M\,\mathrm{d}k \\
&= \int_{K_M} \int_K e^{-(\chi+\rho_L)\eta(e^X k)-\rho_Q\eta(e^X k k_M^{-1})}\,\mathrm{d}k\,\mathrm{d}k_M \\
&= \int_K e^{-(\chi+\rho)\eta(e^X k)}\,\mathrm{d}k = \Xi_\chi^G(e^X).
\end{aligned}$$

Feeding back to (6.6), we obtain the upper bound on the integral term in the LHS of (6.3) by

$$C_1 e^{2\rho\kappa(g)} \int_{\kappa(g)+\mathfrak{a}(r)} \Xi_\chi^G(e^X)\,\mathrm{d}X.$$

For any $X \in \kappa(g)+\mathfrak{a}(r)$, there exist some $Y \in \mathfrak{a}(r)$ and $k' \in K$ for which $\eta(e^X k) = \eta(e^Y k') + \eta(e^{\kappa(g)}k)$. The expression and the K-bi-invariance of the spherical function yield that uniformly for all $X \in \kappa(g)+\mathfrak{a}(r)$,

$$\Xi_\chi^G(e^X) \le C_2\,\Xi_\chi^G(e^{\kappa(g)}) = C_2\,\Xi_\chi^G(g),$$

whence we obtain the inequality (6.3). □

Proposition 6.9. *Let $\chi_0 = (2\beta_{G/H}-1)^+\rho$. Then for any functions $f_1, f_2 \in C_c(G/H)$, there exists a constant $C>0$ such that we have uniformly for all $g \in G$,*

$$\big|\langle \lambda_{G/H}(g) f_1, f_2\rangle\big| \le C\,\Xi_{\chi_0}^G(g).$$

Proof. Since $\lambda_{G/H} = \mathrm{Ind}_Q^G \lambda_{Q/H}$ (Lemma 3.4), we can bound the coefficient on the left hand side from above by passing from f_i to certain positive functions of the form $\mathrm{I}_Q^G(\varphi_i,\xi_i)$, with $\varphi_1,\varphi_2 \in C_c(G)$ and $\xi_1,\xi_2 \in C_c(Q/H)$. Then Lemma 5.1 and Lemma 3.2 yield for some given compactum $B \Subset G$ and uniformly for all $g \in G$ that

$$\big|\langle \lambda_{G/H}(g) f_1, f_2\rangle\big| \le C_1 e^{-2\rho\kappa(g)} \int_{Q\cap BgB} \big|\langle \xi_1, \lambda_{Q/H}(q^{-1})\xi_2\rangle\big|\,\mathrm{d}\nu_Q(q).$$

By Corollary 6.7 we have $\big|\langle \xi_1, \lambda_{Q/H}(q^{-1})\xi_2\rangle\big| \le C_2 \sum_{w\in W_G} \Xi_{w\chi}^L(l_q^{-1})$, whence uniformly for $q \in Q$,

$$\big|\langle \lambda_{G/H}(g) f_1, f_2\rangle\big| \le C_3 e^{-2\rho\kappa(g)} \int_{Q\cap BgB} \sum_{w\in W_G} \Xi_{w\chi_0}^L(l_q^{-1})\, e^{\rho_Q\eta(q)}\,\mathrm{d}q,$$

where we note that the symmetric measure $\mathrm{d}\nu_Q(q) = e^{\rho_Q\eta(q)}\,\mathrm{d}q$. Then Lemma 6.8 yields uniformly for all $g \in G$,

$$\big|\langle \lambda_{G/H}(g) f_1, f_2\rangle\big| \le C_4\,\Xi_\chi^G(g),$$

where we use the fact that $\Xi_{w\chi}^G = \Xi_\chi^G$ for any $w \in W_G$ (Lemma 2.10). □

As a result, we obtain an upper bound of the decay exponent $\theta_{G/H}$, and by combining the lower bound obtained in the previous section we have the following result.

Proposition 6.10. *For any algebraic subgroup H of G, we have*

$$\theta_{G/H} = \max\big\{\beta_{G/H},\, 1/2\big\}.$$

Proof. By Proposition 6.9 and Lemma 2.11, for any $\varepsilon > 0$ and any functions $f_1, f_2 \in C_c(G/H)$, we have the uniform decay for $g \in G$,

$$\left|\left\langle \lambda_{G/H}(g) f_1, f_2 \right\rangle\right| \leq C \exp\left\{-2\left(1 - \varepsilon - \max\left\{\beta_{G/H}, 1/2\right\}\right)\rho\kappa(g)\right\}.$$

By the definition of $\theta_{G/H}$, we get $\theta_{G/H} \leq \max\left\{\beta_{G/H}, 1/2\right\}$. We conclude thanks to Proposition 5.6 which states that $\beta_{G/H} \leq \theta_{G/H}$. □

6.4 Uniform and local decay exponents

In this last part, we complete the proof of Proposition E. We start with some definitions and tools from representation theory.

Definition 6.11. Given $p \in [1, \infty)$, a unitary representation $(\pi, \mathcal{V})$ of G is

- *strongly* L^p if $\langle \pi(\cdot)v, v\rangle \in \mathrm{L}^p(G)$ for a dense subspace of v;
- *almost* L^p if it is strongly $\mathrm{L}^{p+\varepsilon}$ for all $\varepsilon > 0$;
- *totally* L^p if all matrix coefficients of π are L^p;
- *totally* L^{p+} if it is totally $\mathrm{L}^{p+\varepsilon}$ for all $\varepsilon > 0$.

Remark. In the literature, the terms *strongly* L^p and *almost* L^p are widely used (e.g. [10, 30]), though their precise meanings vary across authors; being *totally* L^p often appears in the guise of $A_\pi \subset \mathrm{L}^p(G)$ (e.g. [9, 29]).

Definition 6.12. Given a unitary representation π of G, define $p(\pi)$ to be the minimal $p \geq 1$ such that π is totally L^{p+}. If such p does not exist (e.g. if $\pi = 1_G$ with G noncompact), we put $p(\pi) = \infty$.

Remark. For G noncompact, we always have $p(\pi) \geq 2$ for any unitary representation π. For semisimple G, we have $p(\lambda_{G/H}) \geq p_{G/H}$ by definition, but in general it can happen that $p(\lambda_{G/H}) = 2 > p_{G/H}$, e.g. if H is compact.

Recall from Section 3.2 that $\sigma \prec \pi$ denotes that σ is weakly contained in π as unitary representations of G.

Lemma 6.13. *If $\sigma \prec \pi$ and π is totally L^p, then σ is also totally L^p. As a result, $p(\sigma) \leq p(\pi)$.*

Proof. By [21, Lem 27], π being totally L^p is equivalent to the existence of a constant $C_p > 0$ such that $\|\pi(f)\|_{\mathrm{op}} \leq C_p \|f\|_{p'}$ for all $f \in C_c(G)$, where $1/p + 1/p' = 1$. But $\sigma \prec \pi$ implies $\|\sigma(f)\|_{\mathrm{op}} \leq \|\pi(f)\|_{\mathrm{op}}$ (Fact 3.6), whence $\|\sigma(f)\|_{\mathrm{op}} \leq C_p \|f\|_{p'}$ for all $f \in C_c(G)$, i.e. σ is totally L^p. □

Lemma 6.14. *Let G be a noncompact real semisimple algebraic group. For any closed subgroups $H_2 < H_1 < G$, we have $\theta_{G/H_2} \leq \theta_{G/H_1}$.*

Proof. Herz's *principe de majoration*, cf. [9, Lem 1.4], [5, §3.1]. □

Lemma 6.15 ([29, Thm 5.3]). *For a noncompact real semisimple algebraic group G and $2 \leq p < \infty$, a unitary representation being almost L^p is equivalent to being totally L^{p+}. Hence, for any closed subgroup H,*

(1) if $p_{G/H} \geq 2$, then $p_{G/H} = p(\lambda_{G/H})$;

(2) if $p(\lambda_{G/H}) > 2$, then $p_{G/H} = p(\lambda_{G/H})$. □

Proposition 6.16. *Let G be a noncompact real semisimple algebraic group. For any closed subgroups $H_2 < H_1 < G$ with $\mathrm{L}^2(G/H_1)$ non-tempered, if H_2 is co-amenable in H_1 (meaning $1_{H_1} \prec_{H_1} \lambda_{H_1/H_2}$), then $\theta_{G/H_1} = \theta_{G/H_2}$. In particular, $\delta_{G/H_1} = \delta_{G/H_2}$.*

Remark. The non-tempered assumption cannot be simply removed. For H_2 trivial and H_1 a minimal parabolic subgroup, the exponents $\theta_{G/H_1} = 1/2$ and $\theta_{G/H_2} = 0$ are different despite co-amenability.

Proof. We have proven $\theta_{G/H_1} \geq \theta_{G/H_2}$ in Lemma 6.14.

For the other direction, if H_2 is co-amenable in H_1, i.e. $1_{H_1} \prec \lambda_{H_1/H_2}$, then by unitary induction from H_1 to G we obtain $\lambda_{G/H_1} \prec \lambda_{G/H_2}$. Thus by Lemma 6.13, we have $p(\lambda_{G/H_1}) \leq p(\lambda_{G/H_2})$. But since $\mathrm{L}^2(G/H_1)$ is non-tempered, Theorem A implies that $p_{G/H_1} > 2$, but then Lemma 6.15 further implies $2 < p_{G/H_1} \leq p(\lambda_{G/H_1}) \leq p(\lambda_{G/H_2}) = p_{G/H_2}$. By the equality $\theta_{G/H} = 1 - 1/p_{G/H}$ (Theorem A), we conclude $\theta_{G/H_1} \leq \theta_{G/H_2}$. □

Finally, we can complete the proof of Proposition E.

Proof of Proposition E. Let H be a closed connected subgroup of the real semisimple algebraic group G. By Chevalley's *théorie des répliques*, there exist two algebraic subgroups H_1, H_2 of G which satisfy $H_1^0 \subset H \subset H_2$ and $\mathfrak{h}_1 = [\mathfrak{h}, \mathfrak{h}] = [\mathfrak{h}_2, \mathfrak{h}_2]$, where H_1^0 is the identity component of H_1 with $[H_1 : H_1^0] < \infty$. Hence we have $\theta_{G/H_1^0} = \theta_{G/H_1}$ and $\beta_{G/H_1^0} = \beta_{G/H_1}$.

First assume that $\mathrm{L}^2(G/H)$ is tempered, so that $\theta_{G/H} \leq 1/2$. As H is co-amenable in H_2, we have $\lambda_{G/H_2} \prec \lambda_{G/H}$ by induction, so $\mathrm{L}^2(G/H_2)$ is also tempered and $\beta_{G/H} \leq \beta_{G/H_2} \leq 1/2$ by Proposition 6.10.

Next assume that $\mathrm{L}^2(G/H)$ is non-tempered. Since H_1 is co-amenable in H_2, we can deduce from Proposition 6.16 and Proposition 6.10 that $\beta_{G/H_1} = \theta_{G/H_1} = \theta_{G/H_2} = \beta_{G/H_2}$. Now since $H_1^0 \subset H \subset H_2$, Lemma 6.14 gives $\theta_{G/H_1} \leq \theta_{G/H} \leq \theta_{G/H_2}$. Moreover, since $\mathfrak{h}_1 \subset \mathfrak{h} \subset \mathfrak{h}_2$, we have $\beta_{G/H_1} \leq \beta_{G/H} \leq \beta_{G/H_2}$. Hence, $\theta_{G/H} = \beta_{G/H}$ above $1/2$. □

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