PERFECTOID RINGS AS THOM SPECTRA

ZHOUHANG MAO

ABSTRACT. The Hopkins–Mahowald theorem realizes the Eilenberg–MacLane spectra $\mathbb{F}_p$ as Thom spectra for all primes p. In this article, we record a known proof of a generalization of the Hopkins–Mahowald theorem, realizing perfect rings k as Thom spectra, and we provide a further generalization by realizing perfectoid rings R as Thom spectra. We also discuss even further generalizations to prisms (A, I) and indicate how to adapt our proofs to Breuil–Kisin case.

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1. Introduction

Since most of our results are p-typical, we fix a prime $p \in \mathbb{N}_{>0}$. We know that $\mathbb{F}_p \cong \mathbb{Z}_p/p$, that is to say, $\mathbb{F}_p$ is the free[1] $\mathbb{Z}_p$-algebra in which $p = 0$. The classical Hopkins–Mahowald theorem extends this to a category of "less linear" algebras in which the addition is not commutative or even associative on the nose, but only up to coherent homotopy, which helps compute topological cyclic homology and some variants closely related to integral p-adic cohomology, cf. [BMS19]. More precisely[2], the ring spectrum $\mathbb{F}_p$ is the free object in the category of $\mathbb{E}_2$-$\mathbb{S}_p^\wedge$-algebras with $p \simeq 0$[3].

The main result of this article is to generalize the Hopkins–Mahowald theorem by replacing $\mathbb{F}_p$ by a *perfectoid ring* R. To do so, we construct a spherical enhancement $W^+(R^\flat) \to R$ of Fontaine's map $\theta : W(R^\flat) \to R$, and then the ring spectrum R is the free object in the category of $\mathbb{E}_2$-$W^+(R^\flat)$-algebras with $\xi \simeq 0$ where ξ generates the kernel of Fontaine's map θ. Similar but easier arguments lead to analogous results for complete discrete valuation rings and complete regular local rings, recovering some results in [KN22] (in a slightly different form). Now we explain in more details.

1.1. Main results.

In order to formulate the Hopkins–Mahowald theorem for perfectoid rings, we need the concept of spherical Witt vectors $W^+(k)$[4] for perfect $\mathbb{F}_p$-algebras k, which we will recall in §2. Here we recall that $\pi_0(W^+(k)) = W(k)$ where $W(k)$ is the ring of Witt vectors, and $W^+(\mathbb{F}_p) \simeq \mathbb{S}_p^\wedge$.

The key observation is that this construction leads to a spherical enhancement of Fontaine's A_{inf} by replacing $W(R^\flat)$ with $W^+(R^\flat)$, which allows us to formulate a version of Hopkins–Mahowald theorem for perfectoid rings. The composite map $W^+(R^\flat) \xrightarrow{\tau_{\leq 0}} W(R^\flat) \xrightarrow{\theta} R$ should be understood as a spherical analogue of Fontaine's map $\theta : W(R^\flat) \to R$, satisfying the universal property:

Theorem 1.1 (Proposition 4.4)**.** *The spherical Fontaine's map $W^+(R^\flat) \to R$ is the initial p-complete pro-thickening of R.*

Now the Hopkins–Mahowald theorem for perfectoid rings is the following:

Theorem 1.2 (Theorem 5.2)**.** *Let R be a perfectoid ring. Then the $\mathbb{E}_2$-$W^+(R^\flat)$-algebra R is the free $\mathbb{E}_2$-algebra along with a null-homotopy $\xi \simeq 0$.*

The motivation to realize $\mathbb{F}_p$ as a free $\mathbb{E}_2$-algebra with $p \simeq 0$ is that it describes a direct "generation-relation" like description with respect to the (p-completed) sphere spectrum $\mathbb{S}_p^\wedge$. Similarly, realization of R as a free $\mathbb{E}_2$-$W^+(R)$-algebra with $\xi \simeq 0$ enables us to relate R more directly to the ring $W^+(R^\flat)$ of spherical Witt vectors, which allows us to deduce "topological" results about these rings. For

[1] A common abuse of terminology. It is not a free algebra, but an initial object.

[2] In the main text, we adopt the language of Thom spectra for technical reasons.

[3] More precisely, a chosen nullhomotopy $p \simeq 0$ is part of data.

[4] It is usually denoted by $\mathbb{S}_{W(k)}$, but we find the notation $W^+(k)$ more reasonable.

example, as a consequence, we can compute the topological Hochschild homology $\mathrm{THH}(R)$ (of a perfectoid ring R) as an $\mathbb{E}_1$-ring spectrum and deduce Bökstedt's periodicity. By [KN, Prop 4.7], as in the proof of Thm 4.1 there, we have

Proposition 1.3. *The (relative) topological Hochschild homology* $\mathrm{THH}(R/W^+(R^\flat)) \simeq R \otimes \Omega S^3$ *as* $\mathbb{E}_1$-$W^+(R^\flat)$*-algebras for any perfectoid ring* R*.*

The proof is similar to the classical computation of the Hochschild homology $\mathrm{HH}(R/W(R^\flat))$, via resolving R by $W(R^\flat)$-CDGAs. We refer to first paragraphs of the proof of [HN20, Thm 1.3.2] for this classical case. As a consequence of Proposition 1.3 and the proof of [KN22, Prop 3.5], we have:

Proposition 1.4. *The (absolute) topological Hochschild homology* $\mathrm{THH}(R)^\wedge_p \simeq R \otimes \Omega S^3$ *as* $\mathbb{E}_1$*-ring spectra.*

By known results on the homology of ΩS^3 (a classical reference is [Bot82]), we deduce Bökstedt's periodicity for perfectoid rings (cf. [BMS19, Thm 6.1]), namely, $\pi_*(\mathrm{THH}(R)^\wedge_p) \cong R[u]$ where u is any generator of $\pi_2(\mathrm{THH}(R)^\wedge_p)$ as a $\pi_0(\mathrm{THH}(R)^\wedge_p)$-module. In fact, our question was motivated by Bökstedt's periodicity for perfectoid rings: we wanted to understand why Bökstedt's periodicity holds.

The same strategy recovers [KN22, Rem 3.4 & Thm 9.1]:

Theorem 1.5 (Theorem 6.7 & Corollary 6.20)**.** *Let A be complete discrete valuation ring of mixed characteristic with residue field k being perfect of characteristic p. Then the* $\mathbb{E}_2$-$W^+(k)[\![u]\!]$*-algebra A is the free* $\mathbb{E}_2$*-algebra along with a null-homotopy* $E \simeq 0$*, where E is the Eisenstein polynomial. More generally, let* $(A, \mathfrak{m})$ *be a complete regular local ring of mixed characteristic with residue field k being perfect of characteristic p. Let* $(a_1, \dots, a_n) \subseteq \mathfrak{m}$ *be a regular sequence which generates the maximal ideal* $\mathfrak{m}$*. Then the* $\mathbb{E}_2$-$W^+(k)[\![u_1, \dots, u_n]\!]$*-algebra A is the free* $\mathbb{E}_2$*-algebra along with a null-homotopy* $\phi \simeq 0$*, where* $A \cong W(k)[\![u_1, \dots, u_n]\!]/\phi$*.*

1.2. **Strategies.** We briefly discuss our strategy to prove these Hopkins–Mahowald type theorems. For all kinds of oriented prisms (A, d) in question, we will construct a connective $\mathbb{E}_\infty$-ring spectrum A^+, such that $\pi_0(A^+) \cong A$. For example, when (A, d) is a perfect prism, namely, A/d is a perfectoid ring, we set $A^+ := W^+((A/d)^\flat)$; when $(A, d) = (W(k)[\![u_1, \dots, u_n]\!], \phi)$ is a prism associated to a complete regular local ring, where $\phi(0, \dots, 0) = p$, we set $A^+ := W^+(k)[\![u_1, \dots, u_n]\!]$[5].

Let $A^+/^{\mathbb{E}_2} d$ denote the free $\mathbb{E}_2$-A^+-algebra with a null-homotopy $d \simeq 0$. Then there is a natural map $A^+/^{\mathbb{E}_2} d \to A/d$ of $\mathbb{E}_2$-A^+-algebras. For some maps $(A, d) \to (B, d)$ of oriented prisms, we could construct a map[6] $A^+ \to B^+$ of $\mathbb{E}_\infty$-ring spectra such that the canonical map $B^+ \otimes^{\mathbb{L}}_{A^+} A \to B$ is an equivalence. In this case, if the Hopkins–Mahowald type theorem holds for (A, d), namely, the map $A^+/^{\mathbb{E}_2} d \to A/d$ is an equivalence, then so does it for (B, d).

[5]In this case, it is not clear whether it depends on the presentation $A \cong W(k)[\![u_1, \dots, u_n]\!]$, so A^+ is an abuse of notation.

[6]As for A^+, we only choose a map $A^+ \to B^+$, which is not necessarily functorial.

On the other hand, if the Hopkins–Mahowald type theorem holds for (B, d), and the map $A \to B$ is a thickening, then by a Nakayama-type lemma, we could deduce that it also holds for (A, d). In particular, in the Breuil–Kisin case (or more generally, the case for complete regular local rings), the Nakayama-type lemma in question is precisely the derived Nakayama's lemma (Lemma 6.5); in the perfectoid case, it is an almost perfect version of Nakayama's lemma (Lemma 5.19).

We denote by $\cdot \otimes^{\mathbb{L}}_R \cdot$ the relative smash (or derived tensor) product, by $D(R)$ the symmetric monoidal ∞-category of an $\mathbb{E}_\infty$-ring R, by CAlg_R the ∞-category of $\mathbb{E}_\infty$-R-algebras, and by $\mathrm{CAlg}^\heartsuit_R$ the ∞-category of discrete commutative R-algebras.

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2. Recollection of spherical Witt vectors

In this section, we will review the definition and some basic properties of spherical Witt vectors. We follow [Lur18a, §5.2] for a deformation-theoretic setup.

Definition 2.1 ([Lur18a, Def 5.2.1])**.** Let A be a connective $\mathbb{E}_\infty$-ring, $I \subseteq \pi_0 A$ a finitely generated ideal, $A_0 := \pi_0(A)/I$ and a map $f_0 : A_0 \to B_0$ of discrete commutative rings. We say that a diagram σ

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & \sigma & \downarrow \\ A_0 & \xrightarrow{f_0} & B_0 \end{array}$$

of connective $\mathbb{E}_\infty$-rings *exhibits f as an A-thickening of f_0* if the following conditions are satisfied:

a) The $\mathbb{E}_\infty$-A-algebra B is I-complete;
b) The diagram σ induces an isomorphism $\pi_0(B)/I\pi_0(B) \to B_0$[7] of commutative rings;
c) Let R be a connective $\mathbb{E}_\infty$-algebra over A which is I-complete. Then the canonical map

$$\mathrm{Map}_{\mathrm{CAlg}_A}(B, R) \to \mathrm{Hom}_{\mathrm{CAlg}^\heartsuit_{A_0}}(B_0, \pi_0(R)/I\pi_0(R))$$

is an equivalence.

[7] This is obtained as follows: σ gives rise to a map $B \otimes^{\mathbb{L}}_A A_0 \to B_0$ of $\mathbb{E}_\infty$-rings. Since B_0 is discrete, this gives rise to a map $\pi_0(B)/I\pi_0(B) \simeq \tau_{\le 0}(B \otimes^{\mathbb{L}}_A A_0) \to B_0$.

In particular, when $A_0 = \mathbb{F}_p$ and B_0 is a perfect $\mathbb{F}_p$-algebra, we know the existence and the uniqueness of the thickening in Definition 2.1, and that σ is a pushout diagram [Lur18a, Thm 5.2.5 & Rem 5.2.2]. In particular, classical and spherical Witt vectors fit into this deformation-theoretic setup.

Example 2.2 (Classical Witt vectors, [Lur18a, Example 5.2.6])*.* In Definition 2.1, let $A = \mathbb{Z}_p$ and $I = (p)$. Then $A_0 = \pi_0(A)/I$ is the finite field $\mathbb{F}_p$ and B_0 a perfect $\mathbb{F}_p$-algebra. Let $B := W(B_0)$ be the ring of Witt vectors. It follows from [Lur18a, Prop 5.2.9] that the commutative diagram

$$\begin{array}{ccc} \mathbb{Z}_p & \longrightarrow & W(B_0) \\ \downarrow & & \downarrow \\ \mathbb{F}_p & \longrightarrow & B_0 \end{array}$$

exhibits the map $\mathbb{Z}_p \to W(B_0)$ as an $\mathbb{Z}_p$-thickening of $\mathbb{F}_p \to B_0$. This deformation theoretic description of Witt vectors is also observed in [Bha17, Example 6.1.6]. See also [Ser79, §II.5, Prop 10] for a classical description of this universal property.

Example 2.3 (Spherical Witt vectors, [Lur18a, Example 5.2.7])*.* In Definition 2.1, $A = \mathbb{S}_p^\wedge$ and $I = (p)$. As in Example 2.2, we assume that B_0 is a perfect $\mathbb{F}_p$-algebra. Then [Lur18a, Thm 5.2.5] gives rise to an $\mathbb{E}_\infty$-$\mathbb{S}_p^\wedge$-algebra B, called the *($\mathbb{E}_\infty$-)ring of spherical Witt vectors*, denoted by $W^+(k)$, which is p-complete and the tensor product $\mathbb{F}_p \otimes^{\mathbb{L}}_{\mathbb{S}_p^\wedge} B \simeq \pi_0(B)/p\pi_0(B)$ is isomorphic to B_0.

Proposition 2.4. *The ring $\pi_0(W^+(k))$ is isomorphic to $W(k)$, the ring of Witt vectors, and $W(k) \simeq W^+(k) \otimes^{\mathbb{L}}_{\mathbb{S}_p^\wedge} \mathbb{Z}_p$ for any perfect $\mathbb{F}_p$-algebra k.*

Proof. First, we have a commutative diagram

$$\begin{array}{ccc} \mathbb{S}_p^\wedge & \longrightarrow & W^+(k) \\ \downarrow & & \\ \mathbb{Z}_p & & \downarrow \\ \downarrow & & \\ \mathbb{F}_p & \longrightarrow & k \end{array}$$

given by Example 2.3. The right vertical map $W^+(k) \to k$ factors through the pushout $W^+(k) \otimes^{\mathbb{L}}_{\mathbb{S}_p^\wedge} \mathbb{Z}_p$ in the category of $\mathbb{E}_\infty$-rings. Note that $\mathbb{S}_p^\wedge$ is a coherent $\mathbb{E}_1$-ring, and $\mathbb{Z}_p \simeq \pi_0(\mathbb{S}_p^\wedge)$ is an almost perfect $\mathbb{S}_p^\wedge$-module by [Lur17, Prop 7.2.4.17], which implies that $W^+(k) \otimes^{\mathbb{L}}_{\mathbb{S}_p^\wedge} \mathbb{Z}_p$ is an almost perfect $W^+(k)$-module, since almost perfectness is stable under base change. By Definition 2.1, $W^+(k)$ is a p-complete $\mathbb{E}_\infty$-$\mathbb{S}_p^\wedge$-algebra, therefore by Proposition A.2, the spectrum $W^+(k) \otimes^{\mathbb{L}}_{\mathbb{S}_p^\wedge} \mathbb{Z}_p$ is p-complete. Now we take $A = \mathbb{S}_p^\wedge$, $A' = \mathbb{Z}_p$, $A_0 = \mathbb{F}_p$, $B = W^+(k)$, $B' = W^+(k) \otimes^{\mathbb{L}}_{\mathbb{S}_p^\wedge}$

$\mathbb{Z}_p$ and $B_0 = k$ in [Lur18a, Rem 5.2.4], we deduce that the lower square

$$\begin{array}{ccc} \mathbb{Z}_p & \longrightarrow & W^+(k) \otimes^{\mathbb{L}}_{\mathbb{S}^\wedge_p} \mathbb{Z}_p \\ \downarrow & & \downarrow \\ \mathbb{F}_p & \longrightarrow & k \end{array}$$

constitutes a commutative diagram of thickening as in Definition 2.1. Then it follows from the uniqueness of thickening [Lur18a, Rem 5.2.2] and Example 2.2 that $W^+(k) \otimes^{\mathbb{L}}_{\mathbb{S}^\wedge_p} \mathbb{Z}_p$ is equivalent to $W(k)$ as $\mathbb{E}_\infty$-$\mathbb{Z}_p$-algebras, which implies that $W(k) \simeq \pi_0(W^+(k))$. □

3. Perfect rings being Thom spectra

This section is devoted to realizating perfect $\mathbb{F}_p$-algebras as Thom spectra. Let R be a connective $\mathbb{E}_\infty$-ring spectrum, X a space and $f : X \to \mathrm{BGL}_1(R)$ a map of spaces. The *Thom spectrum* Mf associated to f is defined to be the colimit of the composite functor $X \to \mathrm{BGL}_1(R) \to D(R)$. If X carries an $\mathbb{E}_n$-algebra structure, and f is assumed to be $\mathbb{E}_n$-monoidal, then the Thom spectrum Mf naturally inherits an $\mathbb{E}_n$-R-algebra structure [AB19, Cor 3.2]. In this case, we will call Mf the $\mathbb{E}_n$*-Thom spectrum* associated to f.

Let k be a perfect $\mathbb{F}_p$-algebra and $u \in \mathrm{GL}_1(W(k))$ an invertible element in $W(k)$.

Remark 3.1. The invertible element $1 - pu$ in $W(k)$ gives rise to a map $S^1 \to \mathrm{BGL}_1(W^+(k))$. Since $W^+(k)$ is an $\mathbb{E}_\infty$-ring spectrum, this map extends to a double loop map $\Omega^2 S^3 \simeq \Omega^2\Sigma^2 S^1 \to \mathrm{BGL}_1(W^+(k))$, which we denote by $f_{k,pu}$.

We note that the choice of $1-pu$ essentially imposes an equation $1-pu \simeq 1$. This could be seen by the fact that taking the colimit along $f_{k,pu}$ is essentially taking the homotopy orbits of the $\Omega^2 S^3$-action, which is somehow "multiplying by" $1 - pu$.

Now we formulate the Hopkins–Mahowald theorem[8] for perfect $\mathbb{F}_p$-algebras:

Theorem 3.2 (Hopkins–Mahowald for k)**.** *The Eilenberg–MacLane spectrum k is the $\mathbb{E}_2$-Thom spectrum associated to the map $f_{k,pu}$.*

We need the following Hopkins–Mahowald's theorem[9], cf. [AB19, Thm 5.1]. See also [KN, Thm A.1].

Theorem 3.3 (Hopkins–Mahowald)**.** *Theorem 3.2 holds for $k = \mathbb{F}_p$.*

In order to prove Theorem 3.2, for technical reasons, we start with the special case that $u \in \mathrm{GL}_1(\mathbb{Z}_p) \subseteq \mathrm{GL}_1(W(k))$ which follows directly from the case that $k = \mathbb{F}_p$.

Lemma 3.4. *Theorem 3.2 is true when $u \in \mathrm{GL}_1(\mathbb{Z}_p) \subseteq \mathrm{GL}_1(W(k))$.*

[8]In the first drafts of this article, we simply took $u = 1$. Later, we realized that it might be easier to introduce u to fix a gap in commutative algebra for technical reasons.

[9]In the usual form, $u = 1$, but the proof of Lemma 3.2 shows that the choice of u is inessential.

Proof. We note that the image of the multiplication map $m_{1-pu} : \mathbb{S}_p^\wedge \to \mathbb{S}_p^\wedge$ given by $1-pu \in \pi_0(\mathbb{S}_p^\wedge) \cong \mathbb{Z}_p$ under the canonical (symmetric monoidal) functor $W^+(k) \otimes^{\mathbb{L}}_{\mathbb{S}_p^\wedge} - : D(\mathbb{S}_p^\wedge) \to D(W^+(k))$ is still a multiplication map $m_{1-pu} : W^+(k) \to W^+(k)$ given by $1-pu \in \pi_0(W^+(k)) \cong W(k)$, and therefore the map $f_{k,pu}$ coincides with the composition map

$$\Omega^2 S^3 \xrightarrow{f_{\mathbb{F}_p,pu}} \mathrm{BGL}_1(\mathbb{S}_p^\wedge) \xrightarrow{W^+(k)\otimes^{\mathbb{L}}_{\mathbb{S}_p^\wedge} -} \mathrm{BGL}_1(W^+(k))$$

Since $Mf_{\mathbb{F}_p,pu} \simeq \mathbb{F}_p$ as $\mathbb{E}_2$-ring spectra by Theorem 3.3,

$$Mf_{k,pu} \simeq W^+(k) \otimes^{\mathbb{L}}_{\mathbb{S}_p^\wedge} Mf_{\mathbb{F}_p,pu} \simeq W^+(k) \otimes^{\mathbb{L}}_{\mathbb{S}_p^\wedge} \mathbb{F}_p \simeq k$$

as $\mathbb{E}_2$-ring spectra, where the first equivalence follows from the fact that the functor $W^+(k) \otimes^{\mathbb{L}}_{\mathbb{S}_p^\wedge} -$ is a left adjoint therefore commutes with colimits and the last equivalence following from Example 2.3. □

To prove Theorem 3.2, it suffices to show that $Mf_{k,pu} \simeq Mf_{k,p}$ holds for all $u \in \mathrm{GL}_1(W(k))$, therefore $Mf_{k,pu} \simeq Mf_{k,p} \simeq k$ by Lemma 3.4. We will base the proof on a universal property of Thom spectra, and the author looks forward to an alternative proof which does not depend on this universal property.

Lemma 3.5 (Prop 4.9 in [AB19] along with the discussions after Lem 4.6)**.** *The $\mathbb{E}_2$-Thom spectrum $Mf_{k,pu}$ satisfies the following universal property: For all $\mathbb{E}_2$-$W^+(k)$-algebras A, the mapping space $\mathrm{Map}_{\mathrm{Alg}^{\mathbb{E}_2}_{W^+(k)}}(Mf_{k,pu}, A)$ could be naturally identified with the space of null-homotopies of the composite map $W^+(k) \xrightarrow{m_{pu}} W^+(k) \xrightarrow{\eta} A$ in the category of $W^+(k)$-modules where $\eta : W^+(k) \to A$ is the canonical map given by the $\mathbb{E}_2$-$W^+(k)$-algebra structure, and $m_{pu} : W^+(k) \to W^+(k)$ is the multiplication map given by $pu \in W(k) = \pi_0(W^+(k))$.*

Proof of Theorem 3.2 Note that the multiplication map $m_u : W^+(k) \to W^+(k)$ is an equivalence of $W^+(k)$-modules since $u \in W(k) = \pi_0(W^+(k))$ is invertible. Hence by Lemma 3.5 and that $m_{pu} \simeq m_p \circ m_u$, the map m_u induces an equivalence

$$\begin{aligned} \mathrm{Map}_{\mathrm{Alg}^{\mathbb{E}_2}_{W^+(k)}}(Mf_{k,p}, A) &\longrightarrow \mathrm{Map}_{\mathrm{Alg}^{\mathbb{E}_2}_{W^+(k)}}(Mf_{k,pu}, A) \\ (\eta \circ m_p \simeq 0) &\longmapsto (\eta \circ m_p \circ m_u \simeq 0 \circ m_u = 0) \end{aligned}$$

of spaces which is natural in A. By the Yoneda lemma, we deduce that $Mf_{k,pu} \simeq Mf_{k,p}$ as $\mathbb{E}_2$-$W^+(k)$-algebras. □

4. Recollection of perfectoid rings

In this section, we review the definition and some properties of perfectoid rings.

4.1. **Basic definitions and properties.** Let A be an $\mathbb{F}_p$-algebra, and $\varphi : A \to A$ the Frobenius map. The *(filtered) colimit perfection* A_{perf} of A is the colimit of the telescope $A \xrightarrow{\varphi} A \xrightarrow{\varphi} A \xrightarrow{\varphi} \cdots$. An $\mathbb{F}_p$-algebra A is called *semiperfect* if the map $\varphi : A \to A$ is surjective. For a semiperfect $\mathbb{F}_p$-algebra A, the colimit perfection A_{perf} coincides with $A_{\text{red}} = A/\sqrt{0}$.

There is also a limit perfection. Let R be a commutative ring which is p-adically complete. The *tilt* of R, denoted by $R^\flat$, is a perfect $\mathbb{F}_p$-algebra defined by the limit of the tower

$$\cdots \xrightarrow{\varphi} R/p \xrightarrow{\varphi} R/p \xrightarrow{\varphi} R/p$$

where $\varphi : R/p \to R/p$ is the Frobenius map. In particular, if R is an $\mathbb{F}_p$-algebra, then $R^\flat$ is the *(cofiltered) limit perfection* of R, and if furthermore R is semiperfect, then the canonical map $R^\flat \to R$ is a surjection.

We now construct *Fontaine's map* $\theta : W(R^\flat) \to R$ for every derived p-complete ring R via deformation theory reviewed in §2. One can find a more elementary construction in the special case[10] that R is p-adically complete in, say, [HN20, §1.3].

Let R be a derived p-complete ring. We examine the thickening square

$$\begin{array}{ccc} \mathbb{Z}_p & \longrightarrow & W(R^\flat) \\ \downarrow & & \downarrow \\ \mathbb{F}_p & \longrightarrow & R^\flat \end{array}$$

reviewed in Example 2.2. It follows from the definition of thickening square to the p-complete $\mathbb{E}_\infty$-ring R that the map

$$\mathrm{Hom}_{\mathrm{CAlg}_{\mathbb{Z}_p}}(W(R^\flat), R) \longrightarrow \mathrm{Hom}_{\mathrm{CAlg}_{\mathbb{F}_p}}(R^\flat, R/p)$$

is an equivalence. In particular, the projection $R^\flat = \lim\left(\cdots \xrightarrow{\varphi} R/p\right) \to R/p$ to the last factor is an element of $\mathrm{Hom}_{\mathrm{CAlg}_{\mathbb{F}_p}}(R^\flat, R/p)$, which corresponds to a surjective map $\theta : W(R^\flat) \to R$ of $\mathbb{Z}_p$-algebras. We will call θ *Fontaine's map*.

A commutative ring R is *perfectoid* [BMS18, Def 3.5] if there exists $\pi \in R$ such that $p \in \pi^p R$, such that the ring R is (π)-adically complete, such that the $\mathbb{F}_p$-algebra R/p is semiperfect, and such that the kernel of $\theta : W(R^\flat) \to R$ is a principal ideal.

Let R be a perfectoid ring. We denote by ξ a generator of the kernel of Fontaine's map $\theta : W(R^\flat) \to R$. The *special fiber*, denoted by κ, is the colimit perfection of R/p, that is to say $\kappa := (R/p)_{\text{perf}} = R/\sqrt{pR}$ since R/p is semiperfect.

Proposition 4.1 ([BMS18, Lemma 3.13]). *Let R be a perfectoid ring. Then the commutative diagram*

$$\begin{array}{ccc} W(R^\flat) & \xrightarrow{\theta} & R \\ \downarrow & & \downarrow \\ W(\kappa) & \xrightarrow{\bmod p} & \kappa \end{array}$$

[10] It is a special case by Proposition A.3.

is homotopy coCartesian. In particular, For any generator $\xi \in \ker\theta$, there exists an invertible element $u \in \mathrm{GL}_1(W(\kappa))$ such that the image of $\xi \in W(R^\flat)$ in $W(\kappa)$ is pu.

Proposition 4.2. *Let R be a perfectoid ring. Then the kernel of the composition $R^\flat \to R/p \to \kappa$ is $\sqrt{\xi R^\flat}$.*

Proof. The kernel of the composition $W(R^\flat) \to R/p \to \kappa$ is $\sqrt{pW(R^\flat) + \xi W(R^\flat)}$ whose image under the canonical map $W(R^\flat) \to R^\flat$ is $\sqrt{\xi R^\flat}$. □

4.2. **Universal properties of Fontaine's map (and a spherical analogue).** The results of this subsection will not be used later. However, it is important that Fontaine's map $\theta : W(R^\flat) \to R$ and its spherical analogue $W^+(R^\flat) \to \tau_{\leq 0}(W^+(R^\flat)) \simeq W(R^\flat) \xrightarrow{\theta} R$ satisfy a universal property, a mixed characteristic "absolute" version of thickenings in Definition 2.1. Here is a "derived" version of [Fon94, Thm 1.2.1].

Proposition 4.3 (cf. [HN20, Prop 1.3.4])**.** *Let R be a perfectoid ring. Then Fontaine's map $\theta : W(R^\flat) \to R$ is initial among surjections $\theta_D : D \to R$ of rings such that*

(1) The ideal $\ker(\theta_D) \subseteq D$ is finitely generated[11];
(2) The ring D is derived[12] $(p, \ker(\theta_D))$-complete.

Since $W(R^\flat)$ is p-torsion free and $\ker(\theta)$-torsion free (the later is [BMS18, Lem 3.10(i)]), it is in fact p-adically and $\ker(\theta)$-adically complete. Now we give a spherical version of Fontaine's universal property:

Proposition 4.4. *Let R be a perfectoid ring. We compose Fontaine's map $\theta : W(R^\flat) \to R$ with the 0th Postnikov section $W^+(R^\flat) \to \tau_{\leq 0}(W^+(R^\flat)) = W(R^\flat)$, obtaining the map $\eta : W^+(R^\flat) \to R$. Then we have*

(1) The $\mathbb{E}_\infty$-$\mathbb{S}_p^\wedge$-algebra $W^+(R^\flat)$ of spherical Witt vectors is $(p, \ker\theta)$-complete.
(2) The map $\eta : W^+(R^\flat) \to R$ is initial among all maps $\eta_D : D \to R$ surjective on π_0 where D is an $\mathbb{E}_\infty$-$\mathbb{S}_p^\wedge$-algebra such that
  (a) The ideal $\ker(\pi_0(\eta_D)) \subseteq \pi_0(D)$ is finitely generated;
  (b) The D-module D is $(p, \ker(\pi_0(\eta_D)))$-complete.

Now we want to establish some computational results about homotopy groups of the ring $W^+(k)$ of spherical Witt vectors of a perfect $\mathbb{F}_p$-algebra k. First, Serre's computations of homotopy groups of spheres imply that the sphere spectrum $\mathbb{S}$ is connective, $\pi_0(\mathbb{S}) = \mathbb{Z}$, and for every $n \in \mathbb{N}_{>0}$, the nth stable homotopy group $\pi_n(\mathbb{S})$ is finite, and in particular, having bounded p-power torsion. It then follows from the Milnor sequence of homotopy groups that

[11]The finiteness is assumed so that the derived completeness is well-behaved.

[12]In [HN20, Prop 1.3.4], they assume $(p, \ker(\theta_D))$-adic completeness, but their proof adapts with minor modifications.

Corollary 4.5. *The p-adic sphere spectrum $\mathbb{S}_p^\wedge$ is connective, $\pi_0(\mathbb{S}_p^\wedge) = \mathbb{Z}_p$ and for all $n \in \mathbb{N}_{>0}$, the nth (stable) homotopy group $\pi_n(\mathbb{S}_p^\wedge)$ is a finite direct sum of finite abelian groups of form $\mathbb{Z}/p^r \cong \mathbb{Z}_p/p^r$ for some positive integer $r \in \mathbb{N}_{>0}$.*

We need the following result announced in [Lur18a, Example 5.2.7]:

Proposition 4.6. *Let k be a perfect $\mathbb{F}_p$-algebra. Then the ring of spherical Witt vectors $W^+(k)$ is a flat $\mathbb{S}_p^\wedge$-module*[13].

Proof. Recall that a connective module M over a connective $\mathbb{E}_\infty$-ring R is flat if and only if $M \otimes_R^{\mathbb{L}} \pi_0(R)$ is a flat $\pi_0(R)$-module[14]. Thus it suffices to show that $W^+(k) \otimes_{\mathbb{S}_p^\wedge}^{\mathbb{L}} \mathbb{Z}_p$ is a flat $\mathbb{Z}_p$-module. But $W^+(k) \otimes_{\mathbb{S}_p^\wedge}^{\mathbb{L}} \mathbb{Z}_p \simeq W(k)$ by Proposition 2.4. The result then follows from the p-torsion freeness of $W(k)$ and that $\mathbb{Z}_p$ is a DVR. □

By definition of flat modules, we have

Corollary 4.7. *Let k be a perfect $\mathbb{F}_p$-algebra. Then the ring of spherical Witt vectors $W^+(k)$ is connective, $\pi_0(W^+(k)) = W(k)$, and for all $n \in \mathbb{N}_{>0}$, the nth (stable) homotopy group $\pi_n(W^+(k))$ is a finite direct sum of $W(k)$-modules of form $W(k)/p^r$.*

The proof of Proposition 4.3 essentially adapts to Proposition 4.4:

Proof of Proposition 4.4 We check two statements one by one:

1. Proposition 4.3 tells us that the discrete ring $\pi_0(W^+(R^\flat)) \cong W(R^\flat)$ is $(p, \ker\theta)$-adically complete, therefore by Proposition A.3, it is $(p, \ker\theta)$-complete. In view of Theorem A.1, it remains to show that for each $n \in \mathbb{N}_{>0}$, the homotopy group $\pi_n(W^+(R^\flat))$ is (derived) $(p, \ker\theta)$-complete as a discrete $W(R^\flat)$-module. However, by Corollary 4.7, we have realized $\pi_n(W^+(R^\flat))$ as a finite direct sum of cofibers of $(p, \ker\theta)$-complete modules, therefore it is $(p, \ker\theta)$-complete.
2. Examine the commutative diagram induced by the map $\eta_D : D \to R$:

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{CAlg}_{\mathbb{S}_p^\wedge}}(W^+(R^\flat), D) & \xrightarrow{\simeq} & \mathrm{Hom}_{\mathrm{CAlg}_{\mathbb{F}_p}^\heartsuit}(R^\flat, \pi_0(D)/p) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathrm{CAlg}_{\mathbb{S}_p^\wedge}}(W^+(R^\flat), R) & \xrightarrow{\simeq} & \mathrm{Hom}_{\mathrm{CAlg}_{\mathbb{F}_p}^\heartsuit}(R^\flat, R/p) \end{array}$$

   By Definition 2.1 and Example 2.3, the horizontal maps are homotopy equivalences. Taking the fiber over the map $\eta : W^+(R^\flat) \to R$, seen as an element of $\pi_0\left(\mathrm{Map}_{\mathrm{CAlg}_{\mathbb{S}_p^\wedge}}(W^+(R^\flat), R)\right)$, and letting $\beta : R^\flat \to R/p$ denote the canonical map, we get that

   the space of maps $f : W^+(R^\flat) \to D$ along with a homotopy $\eta_D \circ f \simeq \eta$

---

[13]Let R be an $\mathbb{E}_\infty$-ring. We say that an R-module M is *flat* [Lur17, Def 7.2.2.10] if the $\pi_0(R)$-module $\pi_0(M)$ is flat, and for every $n \in \mathbb{N}$, the map $\pi_n(R) \otimes_{\pi_0(R)} \pi_0(M) \to \pi_n(M)$ is an isomorphism.

[14]Since a connective module M over a connective $\mathbb{E}_\infty$-ring spectrum R is flat if and only if the functor $(-) \otimes_R^{\mathbb{L}} M : D(R) \to D(R)$ preserves the heart, [Lur17, Prop 7.2.2.15].

is weakly equivalent to

the set of maps $g : R^\flat \to \pi_0(D)/p$ such that $(\pi_0(\eta_D)/p) \circ g = \beta$,

and to see the initiality of η, it suffices to show that the later set is a singleton. Now the result follows from deformation theory[15] and the vanishing of the cotangent complex $\mathbb{L}_{R^\flat/\mathbb{F}_p}$ by the perfectness of $R^\flat$.

□

5. The main theorem

Fix a perfectoid ring R and a generator ξ of Fontaine's map $\theta : W(R^\flat) \to R$. This section is devoted to our Hopkins–Mahowald theorem for perfectoid rings, which generalizes Theorem 3.2.

Remark 5.1. As in Remark 3.1, the invertible element $1 - \xi \in \mathrm{GL}_1(W(R^\flat)) = \pi_1(\mathrm{BGL}_1(W^+(R^\flat)))$ gives rise to a map $S^1 \to \mathrm{BGL}_1(W^+(R^\flat))$ which extends to a double loop map $f_{R,\xi} : \Omega^2 S^3 \to \mathrm{BGL}_1(W^+(R^\flat))$.

Theorem 5.2 (Main Theorem)**.** *The Eilenberg–MacLane spectrum R is the $\mathbb{E}_2$-Thom spectrum associated to the map $f_{R,\xi}$ for any perfectoid ring R.*

We first need a much weaker version which says that the 0th homotopy group of the $\mathbb{E}_2$-Thom spectrum in question, as a ring, is isomorphic to R:

Lemma 5.3. *The 0th homotopy group $\pi_0(Mf_{R,\xi})$ of the Thom spectrum associated to $f_{R,\xi}$ is isomorphic to R for any perfectoid ring R.*

Proof. We mimic a segment of the proof of Theorem A.1 in [KN]. Note that $Mf_{R,\xi}$ is connective, so we have

$$\pi_0(Mf_{R,\xi}) \cong \pi_0(W^+(R^\flat)_{h\Omega^3 S^3}) \cong \pi_0(W^+(R^\flat))_{\pi_0(\Omega^3 S^3)}$$

where the $\pi_0(\Omega^3 S^3) \cong \mathbb{Z}$-action on $\pi_0(W^+(R^\flat)) \cong W(R^\flat)$ is given by multiplication by $1 - \xi$, hence

$$\pi_0(W^+(R^\flat))_{\pi_0(\Omega^3 S^3)} \cong W(R^\flat)/(1 - (1 - \xi)) \cong R$$

□

In view of Lemma 5.3, in order to prove Theorem 5.2, it suffices to show that

Proposition 5.4. *The 0th Postnikov section $t_{R,\xi} : Mf_{R,\xi} \to \tau_{\leq 0} Mf_{R,\xi} \simeq R$, being an $\mathbb{E}_2$-map a priori, is an equivalence of spectra.*

To begin with, we first note that the special case when R is a perfect $\mathbb{F}_p$-algebra is already covered by previous considerations:

Lemma 5.5. *The map $t_{R,\xi}$ in question is an equivalence of spectra when R is a perfect $\mathbb{F}_p$-algebra.*

[15]When $\pi_0(D)$ has bounded p-power torsion, the map $\pi_0(D)/p \twoheadrightarrow R/p$ is derived complete along the kernel. In general, we have to consider $\pi_0(D)/p \twoheadleftarrow \pi_0(D)/^{\mathbb{L}}p \twoheadrightarrow R/p$, where both surjective maps are derived complete, and apply the deformation theory twice.

Proof. Theorem 3.2 tells us that there is an equivalence $Mf_{R,\xi} \to R$. The lemma follows from the fact that R lives in $D_{\leq 0}(W^+(R))$ and that the 0th Postnikov section is essentially unique. □

We first note that both $Mf_{R,\xi}$ and R admit canonical $W^+(R^\flat)$-module structures. Our strategy breaks up into several steps:

1. Prove some finiteness and completeness results of $Mf_{R,\xi}$ and R as $W^+(R^\flat)$-modules;
2. Show that $t_{R,\xi}$ is an equivalence after the base change along $W^+(R^\flat) \to W^+(\kappa)$, and hence an equivalence after a further base change along $W^+(\kappa) \to \kappa$ to the special fiber κ;
3. The composition $W^+(R^\flat) \to W^+(\kappa) \to \kappa$ coincides with the composition $W^+(R^\flat) \to R^\flat \to \kappa$, and a Nakayama-like argument shows that $t_{R,\xi}$ is an equivalence after base change along $W^+(R^\flat) \to R^\flat$;
4. Deduce that $t_{R,\xi}$ is an equivalence by completeness.

To proceed, by Proposition 4.1, we may choose an invertible element $u \in \mathrm{GL}_1(W(\kappa))$ associated to ξ so that the image of ξ in $W(\kappa)$ is pu.

5.1. Finiteness and completeness of $Mf_{R,\xi}$ and R as $W^+(R^\flat)$-modules.

Lemma 5.6. *The $W^+(k)$-module $W(k)$ is almost perfect for any perfect $\mathbb{F}_p$-algebra k.*

Proof. If $k = \mathbb{F}_p$, then $W^+(\mathbb{F}_p) \simeq \mathbb{S}_p^\wedge$ is a coherent $\mathbb{E}_\infty$-ring, and $W(\mathbb{F}_p) \simeq \mathbb{Z}_p \simeq \pi_0(W^+(\mathbb{F}_p))$ is an almost perfect $\mathbb{S}_p^\wedge$-module by [Lur17, Prop 7.2.4.17].

In general, by Proposition 2.4, we have $W(k) \simeq W^+(k) \otimes^{\mathbb{L}}_{\mathbb{S}_p^\wedge} \mathbb{Z}_p$, hence $W(k)$ is almost perfect since almost perfectness is stable under base change. □

Corollary 5.7. *The $W^+(R^\flat)$-module R is almost perfect.*

Proof. The $W^+(R^\flat)$-module R is the cofiber of the multiplication map $m_\xi : W(R^\flat) \to W(R^\flat)$ where the domain and the codomain are almost perfect (Lemma 5.6). The result follows from the fact that almost perfect modules are stable under taking cofibers [Lur17, Prop 7.2.4.11]. □

We need an input from algebraic topology:

Proposition 5.8. *The space $\Omega^2 S^3 \in \mathcal{S}$ is of finite type, i.e. is homotopically equivalent to a CW complex with finitely many cells in each degree.*

Proof. This is essentially due to [Wal65, Thm A and B] and Serre. We first note that, the loop space $\Omega^2 S^3$ is a loop space therefore simple [MP12, Cor 1.4.5]. Now we show that $\Omega^2 S^3$ is of finite type, i.e. homotopy equivalent to a CW-complex with finite skeleta. By [MP12, Thm 4.5.2], it suffices to show that $H_i(\Omega^2 S^3; \mathbb{Z})$ are finitely generated for all $i \in \mathbb{N}_{>0}$. The argument is standard (due to Serre): we know that $H_i(S^3; \mathbb{Z})$ are finitely generated for all $i \in \mathbb{N}$. Applying [tD08, Thm 20.4.1] to the fiber sequence $\Omega S^3 \to * \to S^3$ in $\mathcal{S}$, we deduce that $H_i(\Omega S^3)$ are finitely generated for all $i \in \mathbb{N}$. We apply again [tD08, Thm 20.4.1] to the fiber sequence $\Omega^2 S^3 \to * \to \Omega S^3$, we deduce that $H_i(\Omega^2 S^3)$ are finitely generated. $\square$

Colimits of larger diagrams could be computed by colimits of smaller subdiagrams[16]. More precisely, let $F : \mathcal{I} \to \mathrm{Cat}_\infty$ be a diagram of small ∞-categories, and $\mathcal{J}$ be its colimit. Let $\mathcal{K} \to \mathcal{I}$ be the projection map where $\mathcal{K}$ is the category of elements of F [Lur22, Tag 026J][17]. Then $\mathcal{J}$ could be obtained by inverting coCartesian edges in $\mathcal{K}$ [Lur09, Prop 3.3.4.5], and in particular, $h : \mathcal{K} \to \mathcal{J}$ is cofinal. Let $\mathcal{C}$ be an ∞-category and $G : \mathcal{J} \to \mathcal{C}$ a diagram. Then

$$\operatorname{colim} G \simeq \operatorname{colim} G \circ h \simeq \operatorname{colim}_{i \in \mathcal{I}} \operatorname{colim}_{F(i)} G_i$$

where G_i is the composite functor $F(i) \xrightarrow{\text{can}} \mathcal{J} \xrightarrow{G} \mathcal{C}$ and the second follows from the transitivity of left Kan extension [Lur22, Tag 030U][18].

In particular, let $X \in \mathcal{S}$ be a space, R a connective $\mathbb{E}_\infty$-ring, and $F : X \to D(R)$ a diagram such that for every $x \in X$, the R-module $F(x)$ is finite free. Then

(1) If the space X is homotopically equivalent to a finite CW complex, then the R-module $\operatorname{colim} F$ is generated by finite free R-modules under finite colimits, therefore it is a perfect R-module;
(2) If the space X is of finite type, let X_i be its ith skeleton and $F_i : X_i \to X \to D(R)$ the composite functor, then the canonical map $\operatorname{colim} F_i \to \operatorname{colim} F$ is i-connective, which becomes an equivalence after taking colimit over $i \in \mathbb{N}$. It follows that the R-module $\operatorname{colim} F$ is almost perfect. In particular, we deduce the following lemma.

Lemma 5.9. *The $W^+(R^\flat)$-module $Mf_{R,\xi}$ is almost perfect.*

Corollary 5.10. *The $W^+(R^\flat)$-module* $\operatorname{cofib}(t_{R,\xi})$ *is almost perfect.*

Proof. The subcategory of almost perfect modules are closed under taking cofibers and base changes [Lur17, Prop 7.2.4.11]. The statement then follows from Corollary 5.7 and Lemma 5.9. $\square$

Lemma 5.11. *The spectrum R is p-complete.*

Proof. By definition of perfectoid rings, R is p-adically complete, therefore R is p-complete by Proposition A.3. $\square$

Lemma 5.12. *The spectrum $Mf_{R,\xi}$ is p-complete.*

Proof. We note that $W^+(R^\flat)$ is p-complete by definition of spherical Witt vectors, and $Mf_{R,\xi}$ is almost perfect, therefore p-complete by Proposition A.2. $\square$

Corollary 5.13. *The spectrum* $\operatorname{cofib}(t_{R,\xi})$ *is p-complete.*

Proof. It follows from Corollary 5.10 and Proposition A.2. $\square$

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[16] We learnt this from Denis Nardin.

[17] Or *unstraightening* in [Lur09, §3.2].

[18] This generalizes the Bousfield–Kan formula (cf. [Sha18, Cor 12.3]).

5.2. $t_{R,\xi}$ **is an equivalence after the base change along** $W^+(R^\flat) \to W^+(\kappa)$**.** The proof is similar to that of Theorem 3.2, except that we need to be more careful to identify the maps.

Lemma 5.14. *There is a canonical equivalence* $Mf_{\kappa,pu} \xrightarrow{\simeq} W^+(\kappa) \otimes^{\mathbb{L}}_{W^+(R^\flat)} Mf_{R,\xi}$ *of* $W^+(\kappa)$*-modules.*

Proof. We first note that the image of the multiplication map $m_{1-\xi} : W^+(R^\flat) \to W^+(R^\flat)$ under the base change functor $W^+(\kappa) \otimes^{\mathbb{L}}_{W^+(R^\flat)} - : D(W^+(R^\flat)) \to D(W^+(\kappa))$ is the multiplication map $m_{1-pu} : W^+(\kappa) \to W^+(\kappa)$.

Therefore $f_{\kappa,pu}$ coincides with the composite map

$$\Omega^2 S^3 \xrightarrow{f_{R,\xi}} \mathrm{BGL}_1(W^+(R^\flat)) \xrightarrow{W^+(\kappa)\otimes^{\mathbb{L}}_{W^+(R^\flat)} -} \mathrm{BGL}_1(W^+(\kappa))$$

Along with the fact that the functor $W^+(\kappa) \otimes^{\mathbb{L}}_{W^+(R^\flat)} - : D(W^+(R^\flat)) \to D(W^+(\kappa))$ commutes with small colimits, or to be more precise, that the natural transformation $\mathrm{colim}_i (W^+(\kappa) \otimes^{\mathbb{L}}_{W^+(R^\flat)} M_i) \to W^+(\kappa) \otimes^{\mathbb{L}}_{W^+(R^\flat)} (\mathrm{colim}_i M_i)$ is an equivalence for any diagram $(M_i)_i$ in $D(W^+(R^\flat))$, we deduce that there is a canonical equivalence $Mf_{\kappa,pu} \xrightarrow{\simeq} W^+(\kappa) \otimes^{\mathbb{L}}_{W^+(R^\flat)} Mf_{R,\xi}$ as $W^+(\kappa)$-modules. □

It follows directly from Proposition 2.4 that

Lemma 5.15. *Given a morphism of perfect* $\mathbb{F}_p$*-algebras* $k \to K$*, the commutative diagram of* $\mathbb{E}_\infty$*-rings*

$$\begin{array}{ccc} W^+(k) & \longrightarrow & W^+(K) \\ \downarrow & & \downarrow \\ W(k) & \longrightarrow & W(K) \end{array}$$

is a pushout square.

Lemma 5.16. *There is a canonical equivalence* $W^+(\kappa) \otimes^{\mathbb{L}}_{W^+(R^\flat)} R \to \kappa$ *of* $W^+(\kappa)$*-modules.*

Proof. Combining two pushout squares in the category of $\mathbb{E}_\infty$-rings:

$$\begin{array}{ccc} W^+(R^\flat) & \longrightarrow & W^+(\kappa) \\ \downarrow & \sigma & \downarrow \\ W(R^\flat) & \longrightarrow & W(\kappa) \\ \downarrow & \tau & \downarrow \\ R & \longrightarrow & \kappa \end{array}$$

where σ is a pushout square by Lemma 5.15 and τ is a pushout square by Proposition 4.1. □

Lemma 5.17. *The map*

$$W^+(\kappa) \otimes^{\mathbb{L}}_{W^+(R^\flat)} t_{R,\xi} : W^+(\kappa) \otimes^{\mathbb{L}}_{W^+(R^\flat)} Mf_{R,\xi} \to W^+(\kappa) \otimes^{\mathbb{L}}_{W^+(R^\flat)} R$$

is equivalent to the map $t_{\kappa,pu} : Mf_{\kappa,pu} \to \kappa$*.*

Proof. In view of Lemma 5.14 and Lemma 5.16, we only need to show that $t_{\kappa,pu} : Mf_{\kappa,pu} \to \kappa$ coincides with the composition of the equivalences $Mf_{\kappa,pu} \to W^+(\kappa) \otimes^{\mathbb{L}}_{W^+(R^\flat)} Mf_{R,\xi}$, $W^+(\kappa) \otimes^{\mathbb{L}}_{W^+(R^\flat)} t_{R,\xi}$ and $W^+(\kappa) \otimes^{\mathbb{L}}_{W^+(R^\flat)} R \to \kappa$. In other words, it suffices to show that the composition in question is the 0th Postnikov section. We only need to check that the composition induces an isomorphism on π_0 by basic properties of t-structures, since $\tau_{\leq 0} Mf_{\kappa,pu} \simeq \kappa$. It suffices to show that $W^+(\kappa) \otimes^{\mathbb{L}}_{W^+(R^\flat)} t_{R,\xi}$ induces an isomorphism on π_0, and this follows from the fact that all spectra in question are connective and that $t_{R,\xi}$ induces an isomorphism on π_0 by Lemma 5.3. □

Corollary 5.18. *The map $\kappa \otimes^{\mathbb{L}}_{W^+(R^\flat)} t_{R,\xi}$ is an equivalence of spectra.*

Proof. It follows from Lemma 5.17 and Lemma 5.5. □

5.3. $t_{R,\xi}$ is an equivalence after the base change along $W^+(R^\flat) \to R^\flat$.

Lemma 5.19 (Nakayama)**.** *Let A be a connective $\mathbb{E}_\infty$-ring and M an almost perfect A-module. Suppose that $M \otimes^{\mathbb{L}}_A (\pi_0(A)/\operatorname{Rad}(\pi_0(A)))$ is contractible, where $\operatorname{Rad}(\pi_0(A)) \subseteq \pi_0(A)$ is the Jacobson radical. Then so is M itself.*

Proof. Set $B := \pi_0(A)/\operatorname{Rad}(\pi_0(A))$. We show inductively on n that $\pi_n(M) = 0$.

- Since M is bounded below, $\pi_n(M) = 0$ for $n \ll 0$;
- Suppose that for $m < n$ we have $\pi_m(M) = 0$. Then by unrolling the definition, $\pi_n M$ is a compact object in the category of discrete $\pi_0(A)$-modules, therefore is finitely presented and in particular finitely generated. Now
$$0 = \pi_n(B \otimes^{\mathbb{L}}_A M) = B \otimes_{\pi_0(A)} \pi_n(M)$$
It follows that $\pi_n(M) \cong 0$ by Nakayama's lemma.

□

Corollary 5.20. *The map $R^\flat \otimes^{\mathbb{L}}_{W^+(R^\flat)} t_{R,\xi}$ is an equivalence of spectra.*

Note that
$$\pi_0(R^\flat \otimes^{\mathbb{L}}_{W^+(R^\flat)} Mf_{R,\xi}) = R^\flat \otimes_{W(R^\flat)} \pi_0(Mf_{R,\xi}) = R^\flat/\xi R^\flat$$
and
$$\pi_0(R^\flat \otimes^{\mathbb{L}}_{W^+(R^\flat)} R) = R^\flat \otimes_{W(R^\flat)} R = R^\flat/\xi R^\flat$$
and that $R^\flat \otimes^{\mathbb{L}}_{W^+(R^\flat)} Mf_{R,\xi}$, $R^\flat \otimes^{\mathbb{L}}_{W^+(R^\flat)} R$ are connective $\mathbb{E}_\infty$-rings, we conclude that the homotopy groups of these $\mathbb{E}_\infty$-rings are ξ-torsion groups, which implies that for all $n \in \mathbb{Z}$,
$$\xi^2 \pi_n(\operatorname{cofib}(R^\flat \otimes^{\mathbb{L}}_{W^+(R^\flat)} t_{R,\xi})) = 0$$
In addition, since almost perfectness is stable under base changes, we deduce from Corollary 5.10 that
$$\operatorname{cofib}(R^\flat \otimes^{\mathbb{L}}_{W^+(R^\flat)} t_{R,\xi}) \simeq R^\flat \otimes^{\mathbb{L}}_{W^+(R^\flat)} \operatorname{cofib}(t_{R,\xi})$$

is almost perfect. On the other hand, being the cofiber of a map of connective spectra, it is also connective. Then we invoke Lemma 5.19, Corollary 5.18 and Proposition 4.2 to conclude that $\mathrm{cofib}(R^\flat \otimes^{\mathbb{L}}_{W^+(R^\flat)} t_{R,\xi}) \simeq 0$.

5.4. **Conclude: $t_{R,\xi}$ is an equivalence.** We are now at the final stage to conclude a proof of Proposition 5.4, and consequently, Theorem 5.2.

Proof of Proposition 5.4 We recall that by Example 2.3, there is a pushout square of $\mathbb{E}_\infty$-rings:

$$\begin{array}{ccc} \mathbb{S}_p^\wedge & \longrightarrow & W^+(R^\flat) \\ \downarrow & & \downarrow \\ \mathbb{F}_p & \longrightarrow & R^\flat \end{array}$$

Therefore by Corollary 5.20 we have

$$0 \simeq \mathrm{cofib}(R^\flat \otimes^{\mathbb{L}}_{W^+(R^\flat)} t_{R,\xi}) \simeq R^\flat \otimes^{\mathbb{L}}_{W^+(R^\flat)} \mathrm{cofib}(t_{R,\xi}) \simeq \mathbb{F}_p \otimes^{\mathbb{L}}_{\mathbb{S}_p^\wedge} \mathrm{cofib}(t_{R,\xi})$$

We then invoke Corollary A.4 with Corollary 5.13 to deduce that $\mathrm{cofib}(t_{R,\xi}) \simeq 0$. □

6. Analogues

It is worth to note that in Bhatt and Scholze's recent work [BS19], they introduced the concept of prisms (A, I) which serves as a "non-perfect" version of perfectoid rings. Especially, the category of perfect prisms (A, I) is equivalent to that of perfectoid rings A/I, and given a perfectoid ring R, the corresponding perfect prism is given by $(W(R^\flat), \ker\theta)$. It is interesting to know whether we can generalize our description for general orientable prisms (A, I), that is to say,

Question 1. Let $(A, I = (d))$ be an oriented prism. When can we find an $\mathbb{E}_\infty$-ring spectrum A^+ (which satisfies some hypotheses related to A. A naive guess would be that $\pi_0(A^+) = A$) and a map $\Omega^2 S^3 \to \mathrm{BGL}_1(A^+)$ to which the associated $\mathbb{E}_2$-Thom spectrum (possibly after p-completion) coincides with A/I?

We don't know the answer in this generality. However, we will discuss another special class of prism (related to Breuil–Kisin cohomology) for which an analogue holds. This result is expected by experts, and an essentially equivalent result is established in [KN22, Rem 3.4] (see Corollary 6.8). In this section, we will first recall some basic facts about complete discrete valuation rings, then we will indicate briefly how to adapt our proof above to this special class.

6.1. **Preparations.** Basic definitions and facts about *discrete valuation rings* (abbrev. DVR's) can be found in [Ser79, §II]. We recall the key structure theorem:

Proposition 6.1 ([Ser79, §II.5, Thm 4])**.** *Let $(A, \mathfrak{m})$ be a complete DVR of mixed characteristics $(0, p)$ with residue field k being perfect. Let e be its absolute ramification index. Let $\varpi \in \mathfrak{m}$ be a uniformizer. Then there exists an Eisenstein $W(k)$-polynomial $E(u) \in W(k)[u]$ (that is, a $W(k)$-polynomial $E(u) = u^e + \sum_{j=0}^{e-1} a_j u^j$ such that $p \mid a_j$ for $j = 0, \dots, e-1$ and $p^2 \nmid a_0$, where $W(k)$ is the ring of Witt* *vectors as before) along with an isomorphism* $W(k)[u]/(E(u)) \xrightarrow{\sim} A$ *which maps* u *to the uniformizer* $\varpi \in \mathfrak{m}$.

We fix a complete DVR $(A, \mathfrak{m})$ of mixed characteristics $(0, p)$ with residue field k being perfect, absolute ramification index e and a uniformizer $\varpi \in \mathfrak{m}$. We also fix a choice of an Eisenstein $W(k)$-polynomial $E(u) \in W(k)[u]$ as in Proposition 6.1. We first note that, by modulo (p, u) and completeness, we see that the element $1 - E(u) \in W(k)[\![u]\!]$ is invertible.

Let R be an $\mathbb{E}_\infty$-ring, and let $R[u]$ denote the "single variable polynomial R-algebra", that is, the $\mathbb{E}_\infty$-R-algebra $R \otimes^{\mathbb{L}}_{\mathbb{S}} \mathbb{S}[\mathbb{N}]$. Since $\mathbb{S}[\mathbb{N}]$ is a free $\mathbb{S}$-module, we get

Lemma 6.2. *The R-module $R[u]$ is equivalent to the direct sum $\bigoplus_{j=0}^{\infty} u^j R$, a free R-module. The graded homotopy group $\pi_*(R[u])$, as a (graded-commutative) $\pi_*(R)$-algebra, is equivalent to $\pi_*(R)[u]$, where* $\deg u = 0$.

Now let $R[\![u]\!]$ be the (u)-completion (reviewed in §A) of the $\mathbb{E}_\infty$-R-algebra $R[u]$. Since the (u)-completion is lax symmetric monoidal To study $R[\![u]\!]$, we need some preparations.

Lemma 6.3. *Let $n \in \mathbb{N}$ be a natural number. Let $m_{u^n} : R[u] \to R[u]$ be the multiplication map given by $u^n \in \pi_0(R[u]) = \pi_0(R)[u]$. Then*

(1) There is a canonical map $R[u] \xrightarrow{u \mapsto 0} R$ of $\mathbb{E}_\infty$-R-algebras, which fit into a fiber sequence

$$R[u] \xrightarrow{m_u} R[u] \xrightarrow{u \mapsto 0} R$$

of $R[u]$-modules. More generally, the $R[u]$-module $\operatorname{cofib}(m_{u^n})$ *admits a canonical $\mathbb{E}_\infty$-$R[u]$-algebra structure, which is free (thus flat) over R. We will denote this $\mathbb{E}_\infty$-$R[u]$-algebra simply by $R[u]/u^n$.*

(2) The $\pi_*(R[u])$-algebra $\pi_*(R[u]/u^n)$ is equivalent to $\pi_*(R[u])/u^n \cong \pi_*(R(k))[u]/u^n$.

Proof. For any space $X \in \mathcal{S}$, we let $X_+ \in \mathcal{S}_*$ denote the pointed discrete space $\{*\} \cup X$. Especially, $\mathbb{N}_+ = \{*\} \cup \mathbb{N}$ and $(\mathbb{N}_{<n})_+ = \{*\} \cup \mathbb{N}_{<n}$. The addition map $\mathbb{N} \to \mathbb{N}, m \mapsto n + m$ induces a map of pointed spaces $\alpha_n : \mathbb{N}_+ \to \mathbb{N}_+$. Note that in the ∞-category $\mathcal{S}$ of spaces, we have a pushout diagram

$$\begin{array}{ccc} \mathbb{N}_+ & \xrightarrow{\alpha_n} & \mathbb{N}_+ \\ \downarrow & & \downarrow \\ \{*\} & \longrightarrow & (\mathbb{N}_{<n})_+ \end{array} .$$

Note that the right vertical map $\mathbb{N}_+ \to (\mathbb{N}_{<n})_+$ is a surjective map of pointed commutative monoids, where the monoidal structure on $(\mathbb{N}_{<n})_+$ is induced by that on $\mathbb{N}_+$, and the map α_n is compatible with the additive $\mathbb{N}_+$-action. Consequently, applying to the composite of colimit-preserving symmetric monoidal functors $\mathcal{S}_* \to \mathrm{Sp} \xrightarrow{\cdot \otimes^{\mathbb{L}}_{\mathbb{S}} R} D(R)$, we see that the cofiber $\operatorname{cofib}(m_{p^u})$ admits a canonical $\mathbb{E}_\infty$-$R[u]$-algebra structure, which is given by the free R-algebra generated by the pointed commutative monoid $(\mathbb{N}_{<n})_+$. In particular, when $n = 1$, this is precisely R. The second statement also follows from this description. $\square$

Proposition 6.4. *The $\mathbb{E}_\infty$-$W^+(k)$-algebra $W^+(k)[\![u]\!]$ is connective, and the $W(k)[u]$-algebra $\pi_0(W^+(k)[\![u]\!])$ is isomorphic to the (u)-adic completion $W(k)[\![u]\!]$ of the polynomial $W(k)$-algebra $W(k)[u]$.*

Proof. [19]Consider the map $W^+(k)[u] \to \lim_n W^+(k)[u]/u^n$ of $\mathbb{E}_\infty$-rings. Since every $\mathbb{E}_\infty$-$W^+(k)[u]$-algebra $W^+(k)[u]/u^n$ has bounded u^∞-torsion thus u-complete, so is the limit $\lim_n W^+(k)[u]/u^n$. Thus this map factors uniquely through the map $W^+(k)[u] \to W^+(k)[\![u]\!]$ of $\mathbb{E}_\infty$-$W^+(k)[u]$-algebras, giving rise to a map

$$W^+(k)[\![u]\!] \longrightarrow \lim_n W^+(k)[u]/u^n$$

of $\mathbb{E}_\infty$-$W^+(k)[u]$-algebras. We now show that this is an equivalence. By conservativity of the forgetful functor, it suffices to see that this is an equivalence of (u)-complete $W^+(k)[u]$-modules, i.e. the canonical map $W^+(k)[u] \to \lim_n W^+(k)[u]/u^n$ of $W^+(k)[u]$-modules exhibits the target as the (u)-completion of the source, but this follows from [Lur18b, Prop 7.3.2.1]. □

The following derived Nakayama's lemma serves as a key tool in our proof:

Lemma 6.5 (cf. [Sta21, Tag 0G1U])**.** *Let M be a $R[u]$-(resp. $R[\![u]\!]$-)module. If the spectrum $R \otimes^{\mathbb{L}}_{R[u]} M$ (resp. $R \otimes^{\mathbb{L}}_{R[\![u]\!]} M$) is contractible, then so is the (u)-completion of the spectrum M. In particular, if $R[u]$-(resp. $R[\![u]\!]$-)module M is supposed to be (u)-complete, then the spectrum M is contractible.*

Proof. We first assume that the spectrum $R \otimes^{\mathbb{L}}_{R[u]} M$ is contractible. In this case, we apply the exact functor $- \otimes^{\mathbb{L}}_{R[u]} M$ to the cofiber sequence

$$R[u] \xrightarrow{m_u} R[u] \to R \tag{1}$$

indicated in Lemma 6.3 obtaining that the base-changed map

$$M \xrightarrow{m_u \otimes^{\mathbb{L}}_{R[u]} M} M$$

is an equivalence of spectra, thus M is (u)-local. By the semi-orthogonal decomposition into (u)-local objects and (u)-complete objects [Lur18b, Prop 7.3.1.4], we see that the (u)-completion of M is contractible.

If, on the other hand, $R \otimes^{\mathbb{L}}_{R[\![u]\!]} M$ is contractible, then to adopt the proof above, it suffices to establish the cofiber sequence

$$R[\![u]\!] \xrightarrow{m_u} R[\![u]\!] \longrightarrow R \tag{2}$$

We apply the (u)-complete functor to the cofiber sequence (1), and note that multiplying u is the zero map on R, therefore R is (u)-complete, which leads to the cofiber sequence (2). The rest of the proof is same as before. □

[19]This argument is suggested by an anonymous referee.

6.2. **The Breuil–Kisin case.** As before, we fix a complete DVR $(A, \mathfrak{m})$ of mixed characteristics $(0, p)$ with residue field k being perfect, absolute ramification index e, a uniformizer $\varpi \in \mathfrak{m}$ and an Eisenstein $W(k)$-polynomial $E(u) \in W(k)[u]$ which induces an isomorphism $W(k)[u]/(E(u)) \xrightarrow{\sim} A, u \mapsto \varpi$ as in Proposition 6.1. As in Remark 3.1 and Remark 5.1, $1 - E(u) \in W(k)[\![u]\!] = \pi_1(\mathrm{BGL}_1(W^+(k)[\![u]\!]))$ gives rise to a map $f_E : \Omega^2 S^3 \to \mathrm{BGL}_1(W^+(k)[\![u]\!])$. The proof of Lemma 5.3 results in the following analogue:

Lemma 6.6. *The zeroth homotopy group of the $\mathbb{E}_2$-Thom spectrum Mf_E associated to the map f_E is isomorphic to the $W(k)$-algebra $W(k)[\![u]\!]/(E(u)) \cong W(k)[u]/(E(u)) \cong A$.*

The $W(k)[u]$-module structure on A gives rise to a $W^+(k)[u]$-module structure on A. Since A is $\mathfrak{m} = (\varpi)$-adically complete, the $W(k)[u]$-module structure on A also gives rise to a $W(k)[\![u]\!]$-module structure on A and consequently a $W^+(k)[\![u]\!]$-module structure on A. We readily check that these structures are compatible, in the sense that the $W^+(k)[u]$-module structure on A coincides with the image of the $W^+(k)[\![u]\!]$-module A under the forgetful functor $D(W^+(k)[\![u]\!]) \to D(W^+(k)[u])$. Here is the Hopkins–Mahowald theorem in the Breuil–Kisin case:

Theorem 6.7. *The truncation map $t_E : Mf_E \to \pi_0(Mf_E) \cong A$ of $\mathbb{E}_2$-$W^+(k)[\![u]\!]$-algebras is an equivalence of spectra. Thus the Eilenberg–MacLane spectrum A is the $\mathbb{E}_2$-Thom spectrum Mf_E associated to the map $f_E : \Omega^2 S^3 \to \mathrm{BGL}_1(W^+(k)[\![u]\!])$.*

This recovers the following result mentioned in [KN22, Rem 3.4]. We omit its proof since it follows from the proof of Lemma 3.4 and 5.14, or a variant of [KN22, Lem A.2].

Corollary 6.8 (cf. [KN22, Lem A.2])**.** *The $\mathbb{E}_2$-A-algebra $A \otimes^{\mathbb{L}}_{W^+(k)[\![u]\!]} A$ is a free $\mathbb{E}_2$-A-algebra on a single generator in degree 1.*

Recall that $E(u) \in W(k)[u]$ is an Eisenstein $W(k)$-polynomial. Let a_0 denote the constant term of $E(u)$. By assumption, $p \mid a_0$ but $p^2 \nmid a_0$. Let $a_0 = pb_0$ where $b_0 \in W(k)$. Since p is not a zero-divisor in $W(k)$, we have $p \nmid b_0$, which implies that the image of b_0 in $W(k)/p \cong k$ is invertible since k is a field. Now since $W(k)$ is p-adically complete, we have $b_0 \in \mathrm{GL}_1(W(k))$.

The strategy to prove Theorem 6.7 is similar to the approach to attack Theorem 5.2. We first show that the base change of the truncation map t_E along the map $W^+(k)[\![u]\!] \to W^+(k)$ coincides with the truncation map t_{k,a_0}, then it follows from Lemma 5.5 that the base changed map $W^+(k) \otimes^{\mathbb{L}}_{W^+(k)[\![u]\!]} t_E \simeq t_{k,a_0}$ is an equivalence of spectra, and by completeness, we deduce that the map t_E is also an equivalence of spectra by Lemma 6.5.

Lemma 6.9. *There is a canonical equivalence $Mf_{k,a_0} \xrightarrow{\simeq} W^+(k) \otimes^{\mathbb{L}}_{W^+(k)[\![u]\!]} Mf_E$ of $W^+(k)$-modules.*

Proof. We will duplicate the proof of Lemma 5.14. The image of the multiplication map $m_{1-E(u)} : W^+(k)[\![u]\!] \to W^+(k)[\![u]\!]$ under the base change functor

$W^+(k) \otimes^{\mathbb{L}}_{W^+(k)[\![u]\!]} - \,:\, D(W^+(k)[\![u]\!]) \to D(W^+(k))$ is the multiplication map $m_{1-a_0} : W^+(k) \to W^+(k)$. Note also that the base change functor is symmetric monoidal. Now we conclude that the map f_{k,a_0} coincides with the composite map

$$\Omega^2 S^3 \xrightarrow{f_E} \mathrm{BGL}_1(W^+(k)[\![u]\!]) \xrightarrow{W^+(k)\otimes^{\mathbb{L}}_{W^+(k)[\![u]\!]}-} \mathrm{BGL}_1(W^+(k))$$

Thus by commuting the colimit and the base-change, we obtain

$$\begin{aligned} Mf_{k,a_0} &= \operatorname{colim}(W^+(k) \otimes^{\mathbb{L}}_{W^+(k)[\![u]\!]} f_E) \\ &\xrightarrow{\simeq} W^+(k) \otimes^{\mathbb{L}}_{W^+(k)[\![u]\!]} \operatorname{colim} f_E \\ &= W^+(k) \otimes^{\mathbb{L}}_{W^+(k)[\![u]\!]} Mf_E \end{aligned}$$

where by abuse of notation, the colimit of the maps f_{k,a_0} (or f_E respectively) are understood as the colimit of the maps f_{k,a_0} (or f_E respectively) composed with the functor $\mathrm{BGL}_1(W^+(k)) \to D(W^+(k))$ (or $\mathrm{BGL}_1(W^+(k)[\![u]\!]) \to D(W^+(k)[\![u]\!])$ respectively) as in the definition of Thom spectra. □

Lemma 6.10. *There is a canonical equivalence* $W^+(k) \otimes^{\mathbb{L}}_{W^+(k)[\![u]\!]} A \xrightarrow{\simeq} k$ *of* $W^+(k)$*-modules.*

Proof. As in the proof of Lemma 6.5, we identify $W^+(k)$ with the cofiber of the multiplication map $m_u : W^+(k)[\![u]\!] \to W^+(k)[\![u]\!]$ which gives us an equivalence

$$W^+(k) \otimes^{\mathbb{L}}_{W^+(k)[\![u]\!]} A \simeq \operatorname{cofib}\left(A \xrightarrow{m_{A,u}} A\right)$$

Now by the definition of the $W^+(k)[\![u]\!]$-module structure on A and that u is not a zero-divisor in A, we have $\operatorname{cofib}(m_{A,u}) = \operatorname{coker}(m_{A,u})$, which is isomorphic to k. Thus we obtain an equivalence $W^+(k) \otimes^{\mathbb{L}}_{W^+(k)[\![u]\!]} A \simeq k$. We can readily check that this equivalence could be described as follows: consider the commutative diagram in the ∞-category of $\mathbb{E}_\infty$-rings

$$\begin{array}{ccc} W^+(k)[\![u]\!] & \longrightarrow & W^+(k) \\ \downarrow & & \downarrow \\ A & \longrightarrow & k \end{array}$$

where the left vertical map is the composite map $W^+(k)[\![u]\!] \to \pi_0(W^+(k)[\![u]\!]) \simeq W(k)[\![u]\!] \xrightarrow{u \mapsto \varpi} A$ (where the first map is the Postnikov section). The commutative diagram induces a map $W^+(k) \otimes^{\mathbb{L}}_{W^+(k)[\![u]\!]} A \to k$ (note that the left hand side is a pushout of $\mathbb{E}_\infty$-rings), which coincides with the equivalence obtained above. □

Lemma 6.11. *The equivalences in Lemma 6.9 and Lemma 6.10 assembles into a commutative diagram:*

$$\begin{array}{ccc} Mf_{k,a_0} & \xrightarrow{t_{k,a_0}} & k \\ \downarrow \simeq & & \uparrow \simeq \\ W^+(k) \otimes^{\mathbb{L}}_{W^+(k)[\![u]\!]} Mf_E & \longrightarrow & W^+(k) \otimes^{\mathbb{L}}_{W^+(k)[\![u]\!]} A \end{array}$$

where the top horizontal map is the 0th Postnikov section t_{k,a_0} defined in Proposition 5.4 and the bottom horizontal map is the base-changed 0th Postnikov section $W^+(k) \otimes^{\mathbb{L}}_{W^+(k)[\![u]\!]} t_E$.

Proof. As in the proof of Lemma 5.17, it suffices to show that the composite map on the 0th homotopy group $\pi_0(Mf_{k,a_0}) \to \pi_0(W^+(k) \otimes^{\mathbb{L}}_{W^+(k)[\![u]\!]} Mf_E) \to \pi_0(W^+(k) \otimes^{\mathbb{L}}_{W^+(k)[\![u]\!]} A) \to k$ is an isomorphism, which follows from an explicit element chasing. □

Combined with Lemma 5.5, we obtain that

Corollary 6.12. *The base-changed map*

$$W^+(k) \otimes^{\mathbb{L}}_{W^+(k)[\![u]\!]} t_E : W^+(k) \otimes^{\mathbb{L}}_{W^+(k)[\![u]\!]} Mf_E \to W^+(k) \otimes^{\mathbb{L}}_{W^+(k)[\![u]\!]} A$$

is an equivalence of $W^+(k)$-modules.

Apply Lemma 6.5 to the cofiber $\operatorname{cofib}(t_E)$, we deduce that

Corollary 6.13. *The map $t_E : Mf_E \to A$ is an equivalence of spectra after (u)-completion.*

As in Lemma 5.11, we deduce from Theorem A.1 that the $W^+(k)[\![u]\!]$-module A is (u)-complete.

Now, given the nontrivial topological input Proposition 5.8, as in Lemma 5.12 and Corollary 5.13, we deduce that the $W^+(k)[\![u]\!]$-module Mf_E is (u)-complete. Consequently, The cofiber $\operatorname{cofib}(t_E)$ is a (u)-complete $W^+(k)[\![u]\!]$-module, and thus the map t_E is an equivalence of spectra by Corollary 6.13. This completes the proof of Theorem 6.7.

6.3. **Complete regular local rings.** Inspired by [KN22, §9], we will provide a Hopkins–Mahowald theorem for complete regular local rings of mixed characteristic. We will show how to modify our proof of Theorem 6.7 to deduce this. Note that this is also a special case of Question 1, by [BS19, Rem 3.11].

We need some preparations. Let $W^+(k)[u_1, \dots, u_n]$ be the "n-variate polynomial $W^+(k)$-algebra", that is, the $\mathbb{E}_\infty$-$W^+(k)$-algebra $W^+(k) \otimes^{\mathbb{L}}_{\mathbb{S}} \mathbb{S}[\mathbb{N}^n]$. Since the space $\mathbb{N}^n$ is endowed with discrete topology, parallel to Lemma 6.2, we have

Lemma 6.14. *As a $W^+(k)$-module, $W^+(k)[u_1, \dots, u_n]$ is equivalent to the direct sum $\bigoplus_{\alpha\in\mathbb{N}^n} u^\alpha W^+(k)$, a free $W^+(k)$-module. The graded homotopy group $\pi_*(W^+(k)[u_1, \dots, u_n])$, as a (graded-commutative) $\pi_*(W^+(k))$-algebra, is equivalent to $\pi_*(W^+(k))[u_1, \dots, u_n]$, where $\deg u_1 = \cdots = \deg u_n = 0$.*

Now let $W^+(k)[\![u_1, \dots, u_n]\!]$ be the $(u_1, \dots, u_n)$-completion of the $\mathbb{E}_\infty$-$W^+(k)$-algebra $W^+(k)[u_1, \dots, u_n]$. By induction on $n \in \mathbb{N}_{>0}$ and argue as in Proposition 6.4, we obtain:

Proposition 6.15. *The $\mathbb{E}_\infty$-$W^+(k)$-algebra $W^+(k)[\![u_1, \dots, u_n]\!]$ is connective. The zeroth homotopy group of $\pi_0(W^+(k)[\![u_1, \dots, u_n]\!])$ is isomorphic to the $(u_1, \dots, u_n)$-adic completion of the polynomial $W(k)$-algebra $W(k)[u_1, \dots, u_n]$, that is, the formal power series $W(k)$-algebra $W(k)[\![u_1, \dots, u_n]\!]$, as $W(k)$-algebras.*

Similarly, argue inductively on $n \in \mathbb{N}_{>0}$ as in Lemma 6.5, we obtain:

Lemma 6.16. *Let M be a $W^+(k)[u_1, \dots, u_n]$-(resp. $W^+(k)[\![u_1, \dots, u_n]\!]$-)module (spectrum). If the spectrum $W^+(k) \otimes^{\mathbb{L}}_{W^+(k)[u_1,\dots,u_n]} M$ (resp. $W^+(k) \otimes^{\mathbb{L}}_{W^+(k)[\![u_1,\dots,u_n]\!]} M$) is contractible, then so is the $(u_1, \dots, u_n)$-completion of the spectrum M. In particular, if $W^+(k)[u_1, \dots, u_n]$-(resp. $W^+(k)[\![u_1, \dots, u_n]\!]$-)module M is assumed to be $(u_1, \dots, u_n)$-complete, then the spectrum M is contractible.*

We note that in these inductive arguments, we heavily depend on the fact that completeness could be checked on generators of the ideal [Lur18b, Cor 7.3.3.3].

Now we are ready to formulate (a slight generalization of) the Hopkins–Mahowald theorem for complete regular local rings. We fix a positive integer $n \in \mathbb{N}_{>0}$, a perfectoid ring R. As in §5, let $\theta : W(R^\flat) \to R$ be Fontaine's pro-infinitesimal thickening. Let $\phi \in W(R^\flat)[\![u_1, \dots, u_n]\!]$ be a formal power series such that $\phi(0, \dots, 0) \in W(R^\flat)$ is a generator of $\ker \theta$. We recall that $\ker \theta$ is principal by definition. We note that the element $1 - \phi(u_1, \dots, u_n) \in W(R^\flat)[\![u_1, \dots, u_n]\!]$ is invertible, since $1 - \phi(0, \dots, 0) \in W(R^\flat)$ is invertible as the ring $W(R^\flat)$ is $\ker \theta$-adically complete. As in Remark 3.1 and Remark 5.1, the element $1 - \phi(u_1, \dots, u_n) \in \mathrm{GL}_1(W(R^\flat)[\![u_1, \dots, u_n]\!])$ gives rise to an $\mathbb{E}_2$-map $f : \Omega^2 S^3 \to \mathrm{BGL}_1(W(R^\flat)[\![u_1, \dots, u_n]\!])$. The proof of Lemma 5.3 results in the following analogue:

Lemma 6.17. *The zeroth homotopy group of the $\mathbb{E}_2$-Thom spectrum Mf associated to the map f is isomorphic to the $W(R^\flat)$-algebra $W(R^\flat)[\![u_1, \dots, u_n]\!]/(\phi(u_1, \dots, u_n))$.*

We now phrase the following variant of the Hopkins–Mahowald theorem:

Theorem 6.18. *The truncation map $t : Mf \to \pi_0(Mf) \cong W(R^\flat)[\![u_1, \dots, u_n]\!]/(\phi(u_1, \dots, u_n))$ of $\mathbb{E}_2$-$W^+(R^\flat)[\![u_1, \dots, u_n]\!]$-algebras is an equivalence of spectra. Thus the Eilenberg–MacLane spectrum $W(R^\flat)[\![u_1, \dots, u_n]\!]/(\phi(u_1, \dots, u_n))$ is the $\mathbb{E}_2$-Thom spectrum Mf associated to the map $f : \Omega^2 S^3 \to \mathrm{BGL}_1(W^+(R^\flat)[\![u_1, \dots, u_n]\!])$.*

The proof is parallel to that of Theorem 6.7, which we will omit. Now let $(A, \mathfrak{m})$ be a complete regular local ring with residue field $k = A/\mathfrak{m}$ being perfect of characteristic p. We also assume that $p \neq 0$ in A. Let $(a_1, \dots, a_n) \subseteq \mathfrak{m}$ be a regular sequence which generates the maximal ideal $\mathfrak{m}$. We need the following form of Cohen's structure theorem[20].

Lemma 6.19 ([KN22, Lem 9.2])**.** *There exists a map $W(k)[\![u_1, \dots, u_n]\!] \to A$ of rings given by $u_i \mapsto a_i$ for $i = 1, \dots, n$, which is surjective with kernel being principal, generated by a formal power series $\phi \in W(k)[\![u_1, \dots, u_n]\!]$ with $\phi(0, \dots, 0) = p$.*

It then follows from Theorem 6.18 by taking $R = k$ and Lemma 6.19 that

[20]In fact, its proof leads to the slightly more general result: let A be a ring and $I \subseteq A$ a finitely generated ideal such that A is derived (p, I)-complete and the "residue" $R := A/I$ is perfectoid. Then there exists a map $W(R^\flat)[\![u_1, \dots, u_n]\!] \to A$ of rings fitting into the context of Theorem 6.18.

Corollary 6.20. *Let $\phi \in W(k)[\![u_1, \ldots, u_n]\!]$ be a power series as described in Lemma 6.19. Let $f : \Omega^2 S^3 \to \mathrm{BGL}_1(W^+(k)[\![u_1, \ldots, u_n]\!])$ be the map given by the element $1 - \phi(u_1, \ldots, u_n) \in \mathrm{GL}_1(W(k)[\![u_1, \ldots, u_n]\!])$. Then the $\mathbb{E}_2$-Thom spectrum Mf associated to the map f is as an $\mathbb{E}_2$-$W^+(k)[\![u_1, \ldots, u_n]\!]$-algebra equivalent to the Eilenberg–MacLane spectrum A of the complete regular local ring A (of mixed characteristic).*

7. Characterizing Thom spectra as quotients of free $\mathbb{E}_2$-algebras

In this section, we will discuss an alternative characterization of Thom spectra which we learn from [AB19]. This characterization[21] will enable us to peel off some redundant restraints in the definition of Thom spectra. We will rephrase Question 1 more broadly, and explain it in the special case of the Breuil–Kisin case.

Let R be an $\mathbb{E}_\infty$-ring. Let $\mathrm{Free}_R^{\mathbb{E}_2}(u)$ be the free $\mathbb{E}_2$-R-algebra on a single generator u in degree 0. Then for all $\mathbb{E}_2$-R-algebra S and elements $x \in \pi_0(S)$, the universal property of free $\mathbb{E}_2$-R-algebras gives rise to a map $\mathrm{Free}_R^{\mathbb{E}_2}(u) \to S$ induced by "$u \mapsto x$". The following theorem explains how to relate Thom spectra and $\mathbb{E}_2$-algebras:

Theorem 7.1 ([AB19, Thm 4.10])**.** *Let R be an $\mathbb{E}_\infty$-ring and $\alpha \in \pi_1(\mathrm{BGL}_1(R)) \cong \mathrm{GL}_1(\pi_0 R)$. Let $q : S^1 \to \mathrm{BGL}_1(R)$ a loop representing $\alpha \in \pi_1(\mathrm{BGL}_1(R))$. Let $f : \Omega^2 S^3 \to \mathrm{BGL}_1(R)$ be the double loop map associated to q. Then the $\mathbb{E}_2$-Thom spectrum Mf associated to the $\mathbb{E}_2$-map f fits into a pushout diagram of $\mathbb{E}_2$-R-algebras:*

$$\begin{array}{ccc} \mathrm{Free}_R^{\mathbb{E}_2}(u) & \longrightarrow & R \\ \downarrow & & \downarrow \\ R & \longrightarrow & Mf \end{array}$$

where the two maps $\mathrm{Free}_R^{\mathbb{E}_2}(u) \to R$ are induced by "$u \mapsto 0 \in \pi_0(R)$" and "$u \mapsto 1 - \alpha \in \pi_0(R)$" respectively.

Theorem 7.1 shows that the Thom spectrum description is equivalent to the pushout-diagram description. However, we note that the pushout-diagram description is more general in the sense that even if $\alpha \in \pi_0 R$ is not invertible, the pushout-diagram description is still valid while we can no longer, at least superficially, give a Thom spectrum description. We can now rephrase Question 1 as follows:

Question 2. Let $(A, I = (d))$ be an oriented prism. When can we find an $\mathbb{E}_\infty$-ring spectrum A^+ (which satisfies some hypotheses related to A. A naive guess would be that $\pi_0(A^+) = A$) so that the Eilenberg–MacLane spectrum A/I as an $\mathbb{E}_2$-A^+-algebra fits into a pushout diagram

$$\begin{array}{ccc} \mathrm{Free}_{A^+}^{\mathbb{E}_2}(u) & \longrightarrow & A^+ \\ \downarrow & & \downarrow \\ A^+ & \longrightarrow & A/I \end{array}$$

[21] We have already used this characterization in Lemma 3.5.

such that two maps $\mathrm{Free}^{\mathbb{E}_2}_{A^+}(u) \to A^+$ are induced by "$u \mapsto 0 \in \pi_0(A^+)$" and "$u \mapsto d \in \pi_0(A^+)$" respectively?

Theorem 7.1 shows that Theorem 5.2 answers this question affirmatively when (A, I) is a perfect prism $(W(R^\flat), \ker\theta)$, with $A^+ := W^+(R^\flat)$. Similarly, Theorem 6.7 answers this question affirmatively when (A, I) is a prism $(W(k)[\![u]\!], (E(u)))$ associated to Breuil–Kisin cohomology where k is a perfect $\mathbb{F}_p$-algebra and $E(u) \in W(k)[u]$ is an Eisenstein polynomial. We translate Theorem 6.7 to the following theorem.

Theorem 7.2. *In the context of Theorem 6.7, the commutative diagram*

$$\begin{array}{ccc} \mathrm{Free}^{\mathbb{E}_2}_{W^+(k)[u]}(v) & \longrightarrow & W^+(k)[u] \\ \downarrow & & \downarrow \\ W^+(k)[u] & \longrightarrow & A \end{array}$$

becomes coCartesian after (u)-completion in the ∞-category of $\mathbb{E}_2$-$W^+(k)[u]$-algebras, where two maps $\mathrm{Free}^{\mathbb{E}_2}_{W^+(k)[u]}(v) \to W^+(k)[u]$ are induced by "$v \mapsto 0 \in \pi_0(W^+(k)[u])$" and "$v \mapsto E(u) \in \pi_0(W^+(k)[u])$" respectively.

Corollary 7.3 ([KN22, Rem 3.4])**.** *The $\mathbb{E}_2$-A-algebra $A \otimes^{\mathbb{L}}_{W^+(k)[u]} A$ is the p-completion of the free $\mathbb{E}_2$-A-algebra on a single generator in degree 1.*

Proof. Note that $E(u)$ vanishes after tensoring A, and that (u)-completion coincides with (p)-completion for A since ϖ^e/p is an invertible element, the result follows. $\square$

Appendix A. Complete modules

We need two notions of completeness in the article. Let A be a ring and $I \subseteq A$ an ideal. An A-module M is called *I-adically complete* if the canonical map from the A-module M to the (inverse) limit of the tower $\cdots \to M/I^nM \to \cdots \to M/I^2M \to M/IM$ of A-modules is an isomorphism (thus for us, it implies I-adic separatedness).

Let R be a connective $\mathbb{E}_\infty$-ring and $I \subseteq \pi_0(R)$ a finitely generated module. Definitions of *(derived) I-completeness* and *(derived) I-completion* could be found in [Lur18b, §7.3]. When I is principal, it is easy to detect I-completeness: an R-module M is (x)-complete if and only if the limit of the tower $\cdots \xrightarrow{x} M \xrightarrow{x} M \xrightarrow{x} M$ is contractible [Lur18b, Cor 7.3.2.2]. In fact, I-completeness can be checked on individual homotopy groups:

Theorem A.1 ([Lur18b, Theorem 7.3.4.1])**.** *Let R be an $\mathbb{E}_\infty$-ring and $I \subseteq \pi_0 R$ be a finitely generated ideal. Then an R-module M is I-complete if for every $k \in \mathbb{Z}$, the R-module $\pi_k(M)$ is I-complete.*

Almost perfect modules inherit the completeness of the ring:

Proposition A.2 ([Lur18b, Prop 7.3.5.7]). *Let R be a connective $\mathbb{E}_\infty$-ring and $I \subseteq \pi_0 R$ a finitely generated ideal. If R is I-complete, then so are all almost perfect R-modules M.*

Classical completeness implies derived completeness:

Proposition A.3 ([Lur18b, Cor 7.3.6.3]). *Let R be a discrete commutative ring and $I \subseteq R$ a finitely generated ideal. Then every I-adically complete discrete R-module M is (derived) I-complete.*

Corollary A.4. *Let X be a bounded below spectrum. If X is p-complete and $\mathbb{F}_p \otimes^{\mathbb{L}}_{\mathbb{S}} X \simeq 0$, then X is contractible.*

Proof. We will show inductively on $n \in \mathbb{Z}$ that $\pi_n X = 0$.

1. Since X is bounded below, $\pi_n X = 0$ for $n \ll 0$.
2. Suppose now that for every $m < n$, we have $\pi_m X = 0$. We will show that $\pi_n X = 0$. In this case, we have $0 = \pi_n(\mathbb{F}_p \otimes^{\mathbb{L}}_{\mathbb{S}} X) \cong \mathbb{F}_p \otimes_{\mathbb{Z}} \pi_n X$. Thus the multiplication map $\pi_n(X) \xrightarrow{p} \pi_n(X)$ is surjective. Theorem A.1 tells us that $\pi_n(X)$ is derived p-complete, thus the limit of the tower $\cdots \xrightarrow{p} \pi_n(X) \xrightarrow{p} \pi_n(X)$ of surjections is contractible, which implies that $\pi_n(X) \cong 0$.

$\square$

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