Perfectoid rings as Thom spectra

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Introduction

In this talk, we will discuss

- the classical Hopkins-Mahowald theorem,
- prisms, perfectoid rings and the site of (perfect) prisms,
- a proof of the Hopkins-Mahowald theorem for perfectoid rings,
- a proof of the Hopkins-Mahowald theorem for complete regular local rings of mixed char,
- applications to topological Hochschild homology.

Background: $\mathbb{E}_n$-algebras, quotients

Recall

- $\mathbb{E}_\infty$-rings, $\mathrm{CAlg} := \mathrm{Alg}_{\mathbb{E}_\infty}(\mathrm{Sp})$
- $\mathbb{E}_n$-A-algebras where $A \in \mathrm{CAlg}$, $\mathrm{Alg}_{\mathbb{E}_n}(\mathrm{LMod}_A)$

Definition. *Let $A \in \mathrm{CAlg}$, $a \in \pi_0(A)$. The $\mathbb{E}_n$-quotient $A/^{\mathbb{E}_n}a$:= the pushout*

$$\begin{array}{ccc} \mathrm{Free}_A^{(n)}(x) & \xrightarrow{x \mapsto a} & A \\ \downarrow{\scriptstyle x \mapsto 0} & & \downarrow \\ A & \longrightarrow & A/^{\mathbb{E}_n}a \end{array}$$

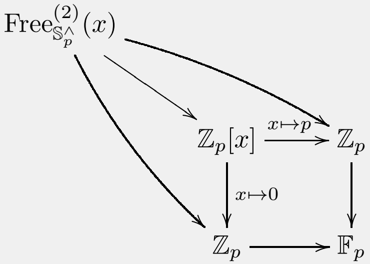
in $\mathrm{Alg}_{\mathbb{E}_n}(\mathrm{LMod}_A)$.

Remark. (Antolín-Camarena-Barthel) A special case of *versal A-algebras* $\Rightarrow$ equivalent to an *$\mathbb{E}_n$-A-Thom spectrum*.

Classical Hopkins-Mahowald Theorem

Notation. $\mathbb{S}_p^\wedge$: *p-complete sphere spectrum*

The canonical map $\mathbb{S}_p^\wedge \to \pi_0(\mathbb{S}_p^\wedge) = \mathbb{Z}_p \to \mathbb{F}_p$ gives rise to a diagram



by definition, get a natural map $\mathbb{S}_p^\wedge /^{\mathbb{E}_2} p \to \mathbb{F}_p$.

Remark. One can show that the induced map $\pi_0(\mathbb{S}_p^\wedge /^{\mathbb{E}_2} p) \to \mathbb{F}_p$ is equivalence $\Rightarrow$ $\mathbb{S}_p^\wedge /^{\mathbb{E}_2} p$ is p-complete.

Theorem. (Hopkins-Mahowald) *The natural map $\mathbb{S}_p^\wedge /^{\mathbb{E}_2} p \to \mathbb{F}_p$ is an equivalence of spectra (therefore also of $\mathbb{E}_2$-$\mathbb{S}_p^\wedge$-algebras).*

Our generalizations: replace $\mathbb{F}_p$ by *perfectoid rings*, or complete regular local rings.

δ-rings and prisms

Definition. (Bhatt-Scholze)

- *A δ-ring A is a ring A along with a derived Frobenius lift, i.e. a ring endomorphism $\varphi : A \to A$ along with a diagram*

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & A \\ \downarrow & & \downarrow \\ A \otimes_{\mathbb{Z}}^{\mathbb{L}} \mathbb{F}_p & \xrightarrow{\mathrm{Frob}} & A \otimes_{\mathbb{Z}}^{\mathbb{L}} \mathbb{F}_p \end{array}$$

  in the ∞-category of simplicial commutative rings (also described by a map $\delta : A \to A$ satisfying several equations).

- *A prism is a pair (A, I) where A is a δ-ring and $I \subseteq A$ is an ideal defining a Cartier divisor in $\mathrm{Spec}(A)$ such that A is (derived) (p, I)-complete and $p \in I + \varphi(I)\,A$.*

- *A prism (A, I) is perfect if the δ-ring A is perfect, i.e. if the Frobenius lift $\varphi : R \to R$ is an automorphism. In this case, I is principal.*

- *A ring S is perfectoid if $\exists$ a perfect prism (A, I) such that $S \cong A/I$ as rings.*

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  in the ∞-category of simplicial commutative rings (also described by a map $\delta : A \to A$ satisfying several equations).

  Remark. *If A is p-torsion free, derived Frobenius lift = Frobenius lift $\varphi : A \to A$ s.t. $\forall x \in A, \varphi(x) \equiv x^p \pmod{p}$.*

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- *A prism* (A, I) *is* perfect *if the* δ*-ring* A *is* perfect*, i.e. if the Frobenius lift* $\varphi : R \to R$ *is an automorphism. In this case,* I *is principal.*
- *A ring* S *is* perfectoid *if* $\exists$ *a perfect prism* (A, I) *such that* $S \cong A/I$ *as rings.*

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- *A ring S is perfectoid if $\exists$ a perfect prism (A, I) such that $S \cong A / I$ as rings.*

Perfect prisms and perfectoid rings

Proposition. (Bhatt-Scholze) *There is a canonical equivalence of categories*

$$\begin{aligned} \{\textit{perfectoid rings } R\} &\simeq \{\textit{perfect prisms } (A,I)\} \\ A/I &\leftarrow\!\shortmid (A,I) \\ S &\mapsto (W(S^\flat), \ker\theta) \end{aligned}$$

where $S^\flat := \lim\left(\cdots \xrightarrow{\text{Frob}} S/p \xrightarrow{\text{Frob}} S/p\right)$ *is the tilt and* $\theta : W(S^\flat) \to S$ *is Fontaine's map.*

Example.

- $(\mathbb{Z}_p, p)$ is a perfect prism.
- More generally, perfect $\mathbb{F}_p$-algebras k are perfectoid $\rightsquigarrow$ perfect prisms $(W(k), p)$.
- **(perfect q-crystalline)** $(\mathbb{Z}[q^{1/p^\infty}]^\wedge_{(p,q-1)}, [p]_q := 1 + q + \cdots + q^{p-1})$

  Remark. A map of prisms $(\mathbb{Z}[q^{1/p^\infty}]^\wedge_{(p,q-1)}, [p]_q) \to (\mathbb{Z}_p, p)$ with kernel $(q-1)$ being complete.
- $(\mathbb{Z}_p[[u]], p-u)$ with $\varphi(u) = u^p$ is a non-perfect prism.
- **(Breuil-Kisin)** $(W(k)[[u]], E(u))$ with $\varphi(u) = u^p$ where $E \in W(k)[u]$ Eisenstein polynomial.

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  Remark. In fact, the map $\mathbb{Z}[q^{1/p^\infty}]^\wedge_{(p,q-1)}/[p]_q \xrightarrow{q \mapsto \mu_{p^\infty}} \mathbb{Z}_p^{\text{cycl}} := (\mathbb{Z}_p[\mu_{p^\infty}])^\wedge_p$ is an equivalence.
- $(\mathbb{Z}_p[[u]], p-u)$ with $\varphi(u) = u^p$ is a non-perfect prism.
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Spherical Witt vectors

Let k be a perfect $\mathbb{F}_p$-algebra. Then

- The cotangent complex $\mathbb{L}_{k/\mathbb{F}_p} \simeq 0$.
- By deformation theory, $\exists$ an equivalence of ∞-categories

$$\{\text{pro-inf-thickenings } R \to \mathbb{F}_p\} \xrightarrow{\simeq} \{\text{pro-inf-thickenings } S \to k\}$$

- **(Lurie)** *The* $\mathbb{E}_\infty$*-ring* $W^+(k)$ *of spherical Witt vectors* is given by

$$(\mathbb{S}_p^\wedge \to \mathbb{F}_p) \longmapsto (W^+(k) \to k)$$

  Remark. $\mathbb{S}_p^\wedge \to \mathbb{F}_p$ is the initial pro-infinitesimal thickening of $\mathbb{F}_p$, therefore so is $W^+(k) \to k$.

- Restrict to discrete rings,

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  Remark. For any map $R \to S$ of rings, the algebraic cotangent complex $\mathbb{L}_{S/R} \neq$ the topological cotangent complex $\mathbb{L}^{\mathrm{top}}_{S/R}$. However, $\mathbb{L}_{S/R} \simeq 0 \iff \mathbb{L}^{\mathrm{top}}_{S/R} \simeq 0$.

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Definition. *A map $R \to S$ of connective $\mathbb{E}_\infty$-rings is called a pro-inf-thickening if it is Adams complete.*

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Spherical Witt vectors

Get a pushout diagram in the ∞-category CAlg

$$\begin{array}{ccc} \mathbb{S}_p^\wedge & \longrightarrow & W^+(k) \\ \big\downarrow & & \big\downarrow \\ \mathbb{Z}_p & \longrightarrow & W(k) \\ \big\downarrow & & \big\downarrow \\ \mathbb{F}_p & \longrightarrow & k \end{array}$$

Corollary. $\pi_0(W^+(k)) \cong W(k)$.

Corollary. *Let* $\mathrm{Perf}_{\mathbb{F}_p}$ *:= the opposite category of perfect* $\mathbb{F}_p$*-algebras. Then* W^+ *is a presheaf of* $\mathbb{E}_\infty$*-rings on* $\mathrm{Perf}_{\mathbb{F}_p}$*, and* $\mathrm{Perf}_{\mathbb{F}_p} \to \mathrm{Mod}, k \mapsto (W^+(k), W(k) \in \mathrm{Mod}_{W^+(k)})$ *is a crystal in almost perfect* W^+*-module spectra.*

Proof of almost perfectness. $\mathbb{S}_p^\wedge$ coherent $\Rightarrow$ $\pi_0(\mathbb{S}_p^\wedge)$ almost perfect / $\mathbb{S}_p^\wedge$ $\Rightarrow$ $W(k)$ almost perfect / $W^+(k)$. $\square$

Hopkins-Mahowald for perfectoid rings

Let S be a perfectoid ring, and ξ a generator of $\ker\Big(W(S^\flat) \xrightarrow{\theta} S\Big)$.

Then the composite map $W^+(S^\flat) \to \pi_0(W^+(S^\flat)) \cong W(S^\flat) \twoheadrightarrow S \rightsquigarrow$ a natural map $W^+(S^\flat)\,/^{\mathbb{E}_2}\xi \to S$.

Our first Hopkins-Mahowald type result is that

Theorem. *The map $W^+(S^\flat)\,/^{\mathbb{E}_2}\xi \to S$ is an equivalence of spectra.*

In particular, when S is a perfect $\mathbb{F}_p$-algebra, we have

Corollary. *Let k be a perfect $\mathbb{F}_p$-algebra. Then the natural map $W^+(k)\,/^{\mathbb{E}_2}p \to k$ is an equivalence of spectra.*

Now we briefly discuss our strategy to prove this.

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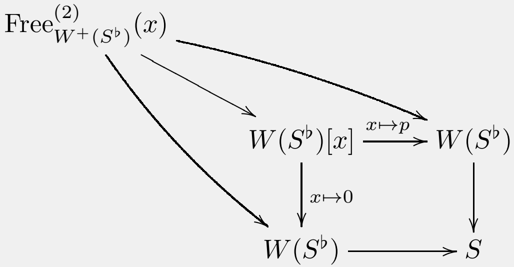

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The site of perfectoid rings

The site Perfd *of perfectoid rings*:

Category. The opposite category of perfectoid rings.

Topology. Covers given by p-*completely faithfully flat* maps $S \to S'$, i.e. $S \otimes_{\mathbb{Z}}^{\mathbb{L}} \mathbb{F}_p \to S' \otimes_{\mathbb{Z}}^{\mathbb{L}} \mathbb{F}_p$ is faithfully flat.

Structure presheaves. $\mathcal{A}_{\mathrm{inf}}: S \mapsto W(S^\flat), \overline{\mathcal{O}}: S \mapsto S, \mathcal{I}_{\mathrm{inf}} := \ker(\mathcal{A}_{\mathrm{inf}} \to \overline{\mathcal{O}})$

Lemma. *The presheaf* $\mathcal{A}_{\mathrm{inf}}$ *is a sheaf of rings, and* $\overline{\mathcal{O}}$ *and* $\mathcal{I}_{\mathrm{inf}}$ *are crystals of perfect* $\mathcal{A}_{\mathrm{inf}}$*-module spectra.*

"Spherical-A_{inf}"

Consider the presheaf $\mathcal{A}^{+}_{\mathrm{inf}}\colon \mathrm{Perfd}^{\mathrm{op}} \to \mathrm{CAlg}^{\wedge}_{p}, S \mapsto W^{+}(S^{\flat})$.

Proposition. *The presheaf $\mathcal{A}^{+}_{\mathrm{inf}}$ is a sheaf, and $\mathcal{A}_{\mathrm{inf}}$ is a crystal in almost perfect $\mathcal{A}^{+}_{\mathrm{inf}}$-module spectra.*

Sheaf:

Lemma. *Given a diagram $S' \leftarrow S \to S''$ in $\mathrm{Perfd}^{\mathrm{op}}$. Then*

1. **(B-S)** $S' \hat{\otimes}^{\mathbb{L}}_{S} S''$ *is a perfectoid ring, and*
2. *the canonical map $\mathcal{A}^{+}_{\mathrm{inf}}(S') \hat{\otimes}^{\mathbb{L}}_{\mathcal{A}^{+}_{\mathrm{inf}}(S)} \mathcal{A}^{+}_{\mathrm{inf}}(S'') \to \mathcal{A}^{+}_{\mathrm{inf}}(S' \hat{\otimes}^{\mathbb{L}}_{S} S'')$ is an equivalence of ($(p, \mathcal{I}_{\mathrm{inf}})$-completed) spectra.*

Lemma. *Let $S \to S'$ be a p-completely faithfully flat map of perfectoid rings. Then $\mathcal{A}^{+}_{\mathrm{inf}}(S) \to \mathcal{A}^{+}_{\mathrm{inf}}(S')$ is $(p, \mathcal{I}_{\mathrm{inf}})$-completely faithfully flat.*

Crystals + almost perfect: seen that W is a crystal in almost perfect W^{+}-module spectra.

Proof of H-M for perfectoid rings

Theorem. (H-M rephrased) *The natural map* $\mathcal{A}^+_{\inf}/^{\mathbb{E}_2}\mathcal{I}_{\inf} \to \overline{\mathcal{O}}$ *of presheaves of spectra on* $\mathrm{Perfd}^{\mathrm{op}}$ *is an equivalence.*

Sketch of proof. Fix $S \in \mathrm{Perfd}$. Suffices to show the equivalence for S.

1. Both sides are crystals in almost perfect $\mathcal{A}^+_{\inf}$-module spectra, and in particular, sheaves.
2. Choose a cover $S \to S' \in \mathrm{Perfd}$ with compatible choice of p-power roots of unity by André's lemma. Thus suffices to show for S'.
3. $\mathbb{Z}_p^{\mathrm{cycl}} := \mathbb{Z}[q^{1/p^\infty}]^\wedge_{(p,q-1)}/[p]_q \to S'$ & crystals $\Rightarrow$ suffices to show for $\mathbb{Z}_p^{\mathrm{cycl}}$.
4. Classical H-M $\Rightarrow$ equiv for $\mathbb{F}_p$, + crystals $\Rightarrow$ equiv for $\mathbb{Z}_p^{\mathrm{cycl}}$ after base change $\mathcal{A}^+_{\inf}(\mathbb{Z}_p^{\mathrm{cycl}}) \to \mathcal{A}^+_{\inf}(\mathbb{F}_p) \simeq \mathbb{S}_p^\wedge$.
5. For $\mathbb{Z}_p^{\mathrm{cycl}}$, both sides are Adams complete w.r.t. $\mathcal{A}^+_{\inf}(\mathbb{Z}_p^{\mathrm{cycl}}) \to \mathcal{A}^+_{\inf}(\mathbb{F}_p) \simeq \mathbb{S}_p^\wedge$.

   Remark. $\pi_0(\mathcal{A}^+_{\inf}(\mathbb{Z}_p^{\mathrm{cycl}})) \cong \mathcal{A}_{\inf}(\mathbb{Z}_p^{\mathrm{cycl}}) \cong \mathbb{Z}[q^{1/p^\infty}]^\wedge_{(p,q-1)}$. □

Generalization to prisms

Let Prism: the site of prisms with flat topology, $\mathrm{Perfd} \subseteq \mathrm{Prism}$ is a full subcategory.

$$\begin{array}{rcl} \mathrm{Shv}(\mathrm{Prism}) & \longrightarrow & \mathrm{Shv}(\mathrm{Perfd}) \\ \mathcal{O} : (A, I) \mapsto A & \longmapsto & \mathcal{A}_{\mathrm{inf}} \\ \overline{\mathcal{O}} : (A, I) \mapsto A/I & \longmapsto & \overline{\mathcal{O}} \\ ? & \longmapsto & \mathcal{A}_{\mathrm{inf}}^{+} \end{array}$$

Question. Does the sheaf $\mathcal{A}_{\mathrm{inf}}^{+} \in \mathrm{Shv}(\mathrm{Perfd})$ (+ the map $\mathcal{A}_{\mathrm{inf}}^{+} \to \mathcal{A}_{\mathrm{inf}}$) extends to some $\mathcal{O}^{+} \in \mathrm{Shv}(\mathrm{Prism})$?

If true, then the previous proof $\Rightarrow$ the natural map $\mathcal{O}^{+} /^{\mathbb{E}_2} \mathcal{I}_{\mathrm{inf}} \to \overline{\mathcal{O}}$ is an equivalence, at least for orientable prisms ((A, I) is *orientable* if I is principal).

We now consider a candidate of values of $\mathcal{O}^{+}$ for some prisms (A, I).

Deforming a perfect prism

Let (A,d): an oriented perfect prism.

Consider $(B := A[[u_1,\dots,u_n]], f)$ where $\varphi(u_i) = u_i^p$ and $f \in B$ s.t. $f(0) = d$.

This is a prism, + a map $B \twoheadrightarrow A, u_i \mapsto 0$ of prisms, seen as a pro-infinitesimal thickening.

> **Example.** A complete regular local ring R of mixed char $(0,p)$ with perfect residue field k,
> Cohen $\Rightarrow$ $R \cong W(k)[[u_1,\dots,u_n]]/E$ where $E \in W(k)[[u_1,\dots,u_n]]$ s.t. $v_p(E(0)) = 1$.
> Get $(A,d) = (W(k), E(0))$ and $f = E$.

A candidate for $\mathcal{O}^+(B,f)$: $\mathcal{A}_{\mathrm{inf}}^+(A/d)[[u_1,\dots,u_n]] =: B^+$.

Theorem. *The natural map $B^+/^{\mathbb{E}_2} f \to B/f$ induced by the composite map $B^+ \to \pi_0(B^+) \cong B \to B/f$ is an equivalence of spectra.*

Proof. Follows from equiv after base change along $B^+ \to A^+$ + both sides are Adams complete w.r.t. $B^+ \to A^+$. $\square$

Computing THH

Application: alternative computations of THH in Bhatt-Morrow-Scholze and Krause-Nikolaus.

Proposition. *Let $A \in \mathrm{CAlg}$, $d \in \pi_0(A)$. Then*

$$\mathrm{THH}((A/^{\mathbb{E}_2} d)/A) \simeq (A/^{\mathbb{E}_2} d) \otimes \Omega S^3$$

as $\mathbb{E}_1$-A-algebras.

Corollary. *Let R: a perfectoid ring. Then*

$$\mathrm{THH}(R/W^+(R^\flat)) \simeq R \otimes \Omega S^3.$$

Corollary. (B-M-S) *Let R: a perfectoid ring. Then $\pi_*(\mathrm{THH}(R)_p^\wedge) \cong R[u]$ where $\deg u = 2$.*

Corollary. *Let $R \cong W(k)[[u_1, \ldots, u_n]]/E$: a compl. reg. loc. ring of mixed char. Then*

$$\mathrm{THH}(R/W^+(k)[[u_1, \ldots, u_n]]) \simeq R \otimes \Omega S^3$$

Corollary. (K-N, B-M-S) *Let $R \cong W(k)[[u_1, \ldots, u_n]]/E$. Then $\pi_*(\mathrm{THH}(R/\mathbb{S}[[u_1, \ldots, u_n]])_p^\wedge) \cong R[u]$ where $\deg u = 2$.*